Chapter 2

Set and probability theory

2.1. Set operation

A set is a collection of objects, which are elements of the set. Sets are usually denoted using capital letters, while elements are denotes using lower-case letters. The elements of a set can be interpreted as the outcomes or the results of an experiment or a random trial.

Sample space The sample space is an exhaustive collection of objects/outcomes of an experiment.

Let S denotes a sample space. Let A and B be two sets of elements that belong to the sample space.

Union The union of two sets A and B, $A \cup B$ is the set of elements which belong to A, to B or both. If $A \cup B = S$, they we say that A and B are mutually exhaustive.

Intersection Consider two sets A and B. The intersection $A \cap B$ is the set of elements common to A and B . A and B are said to be disjoint if the intersection of A and B is an empty set, i.e. if $A \cap B = \emptyset$. We say also that A and B are mutually exclusive.

Subsets We say that A is a subset of B, i.e. $A \subset B$, if all elements of A belong to B. If A is not a subset of B, we write $A \not\subset B$

Complementarity The difference $A \setminus B$ is the set of elements of A which do not belong to S . A complement $\overline{A}$ is defined as

$$\overline{A} = S \setminus A$$

and denotes the set of elements that belong to S outside A.

2.2. Set identities

Below some set relations.

- A set B can be obtained as a union of two disjoint sets, i.e. $B = (A \cap B) \cup (\overline{A} \cap B)$
- $A \cup B = A \cup (\overline{A} \cap B)$
- Distributive law: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- First De Morgan's law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- Second De Morgan's law: $\overline{A \cap B} = \overline{A} \cup \overline{B}$

2.3. Probability

Let S be a sample space. By probability on S we mean a numerical function $P(A)$ of events $A \subset S$ such that the following properties hold:

- *non-negativity*: for any $A \subset S$, i.e. for any event that is in the sample space, $0 \leq P(A) \leq 1$
- Let $A \subset S$, and let $O_i \in A$ for $i = 1, ...k$. Then

$$P(A) = \sum_{i=1}^{k} P(O_i)$$

- *completeness*: $P(S) = 1$

Notice that

- if A and B are mutually exclusive, i.e. $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
- if A and B are not mutually exclusive, i.e. $A \cap B \neq \emptyset$, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- if A and B are collectively exhaustive, i.e. $A \cup B = S$, then $P(A \cup B) = 1$. Therefore, $P(\overline{A}) = P(S \setminus A) = P(S) - P(A) = 1 - P(A)$.

An impossible event can be defined by $P(A) = 0$. When $P(A) = 1$, we say that A is a sure event.

Joint probability The event $A \cap B$ is called a joint event and its probability $P(A \cap B)$ is called a joint probability.

Conditional probability Let $P(B) > 0$. The probability $P(A|B)$ is called probability of A conditional on B and is defined as follows

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

This obviously implies that $P(A \cap B) = P(A|B)P(B)$.

Independence Events A and B are called statistically independent if occurrence of one of them does not influence in any way the chances of occurring of the other, i.e. if $P(A \cup B) = P(A)P(B)$. When $P(B) > 0$, then this is equivalent to $P(A \cup B) = P(A)$

Bayes theorem: $P(A|B) = P(B|A)\frac{P(A)}{P(B)}$

Proof. Using the definition of conditional probability, we can write:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)}\frac{P(A)}{P(B)} = P(B|A)\frac{P(A)}{P(B)}$$

2.4. Exercises

Exercise 1 Out of a group of 20 people, 15 enjoy cycling and 8 enjoy reading. 2 of them enjoy neither cycling nor reading. How many people enjoy both cycling and reading?

Exercise 2 A committee of two members is to be chosen from a group of 4 male and 2 female candidates. If each member has the same probability of being chosen, what is the probability that both members are male?

Exercise 3 The manager of a music store finds that 30% of customers entering the store ask ban assistant for help, and 20% of customers entering the store make a purchase before leaving. Also, 15% of customers entering the store both ask an assistant for help and make a purchase before leaving.

- What is the probability that a customer entering the store will ask an assistant for help or make a purchase or both?
- What is the probability that a customer that asks an assistant for help will make a purchase before leaving?
- Consider the events “asks assistant for help" and “makes purchase"
  - Are these events mutually exclusive?
  - Are they collectively exhaustive?
  - Are they independent?

Exercise 4 Let the sample space be the collection of all possible outcomes of rolling one die. Let A be the event: "# rolled is even". Let B be the event: "# rolled is at least 4". Find the following:

- complement of A
- complement of B
- intersection of A and B
- intersection of A and B
- union of A and B
- union of A and A

Are A and B mutually exclusive? Are A and B collectively exhaustive?