Chapter 4

Continuous random variables

Exercise 1

- To calculate the probability of x we integrate its PDF over the appropriate range of values:

$$P(0 \leq X \leq 1) = \int_0^1 \frac{1}{9}x^2 dx = \left[\frac{x^3}{27}\right]_0^1 = \frac{1}{27} - 0 = \frac{1}{27}$$
$$P(0 \leq X \leq 2) = \int_0^2 \frac{1}{9}x^2 dx = \left[\frac{x^3}{27}\right]_0^2 = \frac{8}{27} - 0 = \frac{8}{27}$$
$$P(1 \leq X \leq 2) = \int_1^2 \frac{1}{9}x^2 dx = \left[\frac{x^3}{27}\right]_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$$

- The expected value of X is:

$$\mathrm{E}[X] = \int_0^3 \frac{1}{9}x^3 dx = \left[\frac{x^4}{36}\right]_0^3 = \frac{3^4}{36} - 0 = \frac{9}{4}$$

- The variance of X is:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

Since:

$$\mathrm{E}[X^2] = \int_0^3 \frac{1}{9}x^4 dx = \left[\frac{x^5}{45}\right]_0^3 = \frac{3^5}{45} - 0 = \frac{243}{45} = \frac{27}{5}$$

Then:

$$\mathrm{VAR}[X] = \frac{27}{5} - \left(\frac{9}{4}\right)^2 = 0.3375$$

Exercise 2 The expected value of X is equal to

$$\mathrm{E}[X] = \int_0^1 x f(x) dx = \int_0^1 3x^3 dx = 3\left[\frac{x^4}{4}\right]_0^1 = 3\left[\frac{1}{4} - 0\right] = \frac{3}{4}$$

The variance of X is equal to

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$$

Since:

$$\mathrm{E}[X^2] = \int_0^1 x^2 f(x) dx = \int_0^1 3x^4 dx = 3\left[\frac{x^5}{5}\right]_0^1 = 3\left[\frac{1}{5} - 0\right] = \frac{3}{5}$$

then:

$$\mathrm{VAR}[X] = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{5} - \frac{9}{16} = 0.0375$$

Exercise 3

- Solution to first question:
  - The following probabilities are obtained from Table 1 of the Statistical Tables booklet:

    $$P(0 \leq Z \leq 1.20) = P(Z \leq 1.20) - P(Z \leq 0) = 0.8849 - 0.5 = 0.3849$$

  - Note that the Statistics Tables only give probabilities for positive Z values, so we cannot find the probability $P(Z < -1.33)$ directly from these tables. But, as the standard normal distribution is symmetric

    $$P(-1.33 \leq Z \leq 0) = P(0 \leq Z \leq 1.33)$$

    which we can calculate as follows

    $$P(0 \leq Z \leq 1.33) = P(Z \leq 1.33) - P(Z \leq 0) = 0.9082 - 0.5 = 0.4082$$

  - $P(Z > 1.33) = 1 - P(Z < 1.33) = 1 - 0.9082 = 0.0918$
  - $P(-0.77 \leq Z \leq 1.68) = P(0 \leq Z \leq 1.68) + P(0 \leq Z \leq 0.77) =$ $(0.9535 - 0.5) + (0.7794 - 0.5) = 0.4535 + 0.2794 = 0.7329$
- Solution to second question: This probability is obtained from Table 2 of the

Statistical Tables booklet. As before, we can write

$$
\begin{aligned}
P(x \leq Z \leq 1.68) = P(Z \leq 1.68) - P(Z \leq x) = 0.2 \\
0.9535 - P(Z \leq x) = 0.2 \\
P(Z \leq x) = 0.9535 - 0.2 = 0.7535
\end{aligned}
$$

Table 2 gives $P(Z > x)$ where

$$P(Z > x) = 1 - P(Z \leq x) = 1 - 0.7535 = 0.2465$$

This value (0.2465) is half way between the values 0.246 and 0.247 listed in Table 2. Therefore:

$$x = 0.5(0.6871) + 0.5(0.6840) = 0.6856$$

Exercise 4 We need to transform the normally distributed variable miles (X) into standard normally distributed random variable Z and use values from Table 1 of the Statistical Tables. We are told that X is normally distributed with a mean of 35000 and a standard deviation of 4000. Therefore:

$$Z = \frac{X - 35000}{4000}$$

is standard normally distributed (it has a mean of 0 and a standard deviation of 1). Applying this standardization technique, for the first question we get:

$$P(35000 < X < 38000) = P\left(\frac{35000 - 35000}{4000} < Z < \frac{38000 - 35000}{4000}\right) = P(0 < Z < 0.75)$$

Therefore:

$$P(0 < Z < 0.75) = P(Z < 0.75) - P(Z < 0) = 0.7734 - 0.5 = 0.2734$$

For the second question, we get:

$$
\begin{aligned}
P(X < 32000) = P\left(Z < \frac{32000 - 35000}{4000}\right) \\
= P(Z < -0.75) = P(Z > 0.75) = 1 - P(Z < 0.75) = 1 - 0.7734 = 0.2266
\end{aligned}
$$

Exercise 5 The cdf of X is

$$P(X \leq z) = \int_0^z f(x)dx = \int_0^z 0.5dx = 0.5[x]_0^z = 0.5z$$

The probability that X takes values between 0.5 and 1.5 is equal to

$$P(0.5 \leq X \leq 1.5) = P(X \leq 1.5) - P(X \leq 0.5) = 0.5(1.5) - 0.5(0.5) = 0.50$$

Exercise 6 By assumption, $X \sim \mathcal{N}(1000000, 30000^2)$. The probability that the portfolio value is between 970000 and 1060000 is equal to

$$P(970000 < X < 1060000) = P\left(\frac{970000 - 1000000}{30000} < Z < \frac{1060000 - 1000000}{30000}\right) =$$
$$P(-1 < Z < 2) = P(Z < 2) - P(Z < -1) = 0.97725 - 0.15866 = 0.81859$$