

Advanced Macroeconomics

Firm dynamics

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Firm dynamics: facts

Definitions

- Establishment/plant: a fixed physical location of economic activity
- Firm: all establishments under control of the enterprise
- Firm size:
 - “Base year”: number of employees n_{it} at start of given period t
 - “Current year”: simple average of number of employees between periods t and $t - 1$
- Firm size growth rate, g_{it} :

$$g_{it} = \frac{n_{it} - n_{it-1}}{X_{it}}$$

where X_{it} is the average firm size in the current year:

$$X_{it} = \frac{n_{it} + n_{it-1}}{2}$$

- Job creation rate

$$JC_t = \frac{\sum_{i:n_{it} > n_{it-1}} (n_{it} - n_{it-1})}{\sum_i n_{it-1}} = \sum_i \frac{X_{it}}{\sum_i X_{it}} \max\{0, g_{it}\}$$

\implies employment gains summed over all establishments that expand or start up between t and $t - 1$

- Job destruction rate

$$JD_t = \frac{\sum_{i:n_{it} < n_{it-1}} (n_{it-1} - n_{it})}{\sum_i n_{it-1}} = \sum_i \frac{X_{it}}{\sum_i X_{it}} \max\{0, -g_{it}\}$$

\implies employment losses summed over all establishments that contract or shut down between t and $t - 1$

- Job reallocation: $JR_t = JC_t + JD_t$

- Let F_{it} be the measure of firms at time t . Let $\mathbf{1}_{it}^e$ and $\mathbf{1}_{it}^x$ be two indicator functions for firm entry and exit at time t .
- Measuring firm entry

$$\text{Entry}_t = \frac{\sum_i F_{it} [\mathbf{1}_{it}^e = 1]}{\sum_i F_{it-1}}$$

- Measuring firm exit

$$\text{Exit}_t = \frac{\sum_i F_{it-1} [\mathbf{1}_{it}^x = 1]}{\sum_i F_{it-1}}$$

- Firm turnover: $\text{Entry}_t + \text{Exit}_t$

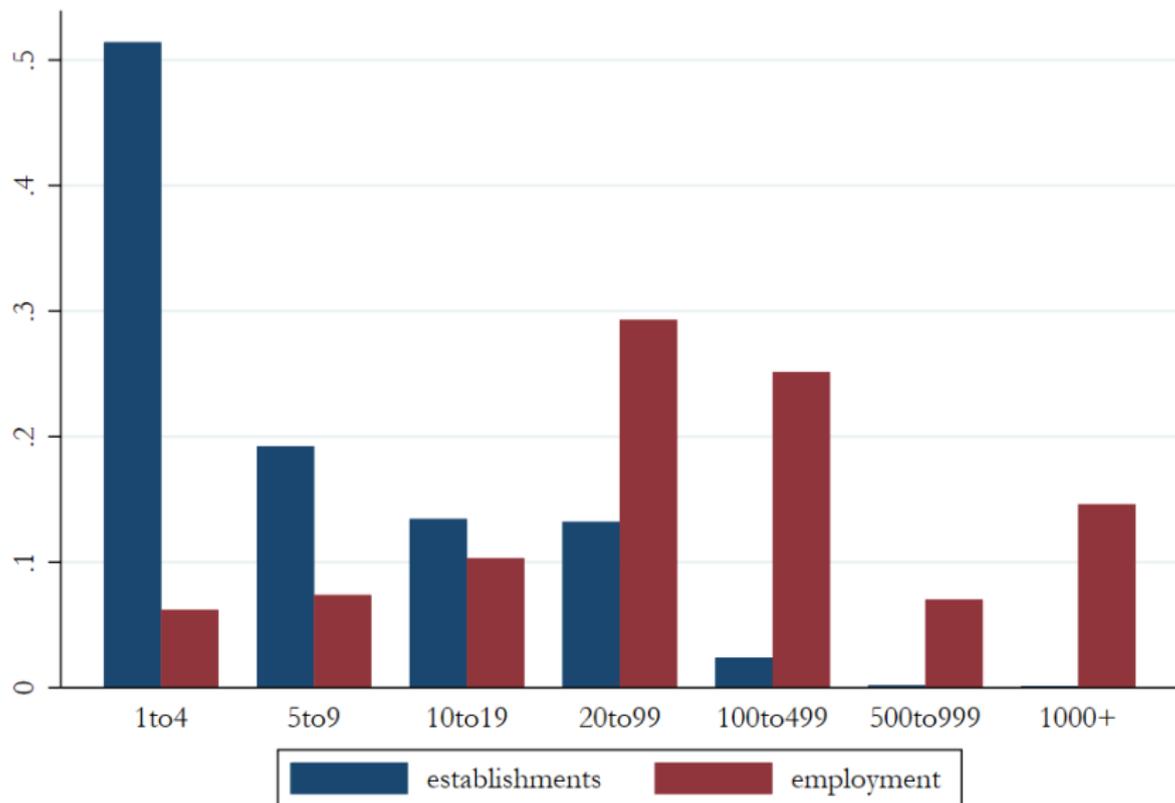
- Business Dynamics Statistics (BDS)
 - It is made from Longitudinal Business Database (LBD) and it is public data. It includes the numbers of firms and establishments, firm age distribution, employment distribution, entry/exit, job creation and job destruction rates.
- Statistics of U.S. Businesses (SUSB)
 - It is public data. It includes annual numbers of firms, establishment, employment, and annual payroll by location (states, MSAs) and industries (3-digit NAICS)
- Quarterly Census of Employment and Wages (QCEW)
 - It is made from the Business Employment Dynamics Dataset (BED) and it is public data. It reports quarterly count of employment and wages reported by employers, available at the county, MSA, state and national levels by industry
- Job Openings and Labor Turnover Survey (JOLTS)
 - Monthly data from a sample of approximately 16,000 U.S. business establishments. Asks job openings (vacancies), hires, separations, quits, layoffs

Firm/establishment demographics

# firms	5.32 mil	# establishments	7.15mil
Average size	24.75	Average size	18.22
		JC rate	11.87%
		JD rate	10.17%
		Entry rate	9.23%
		Exit rate	8.49%

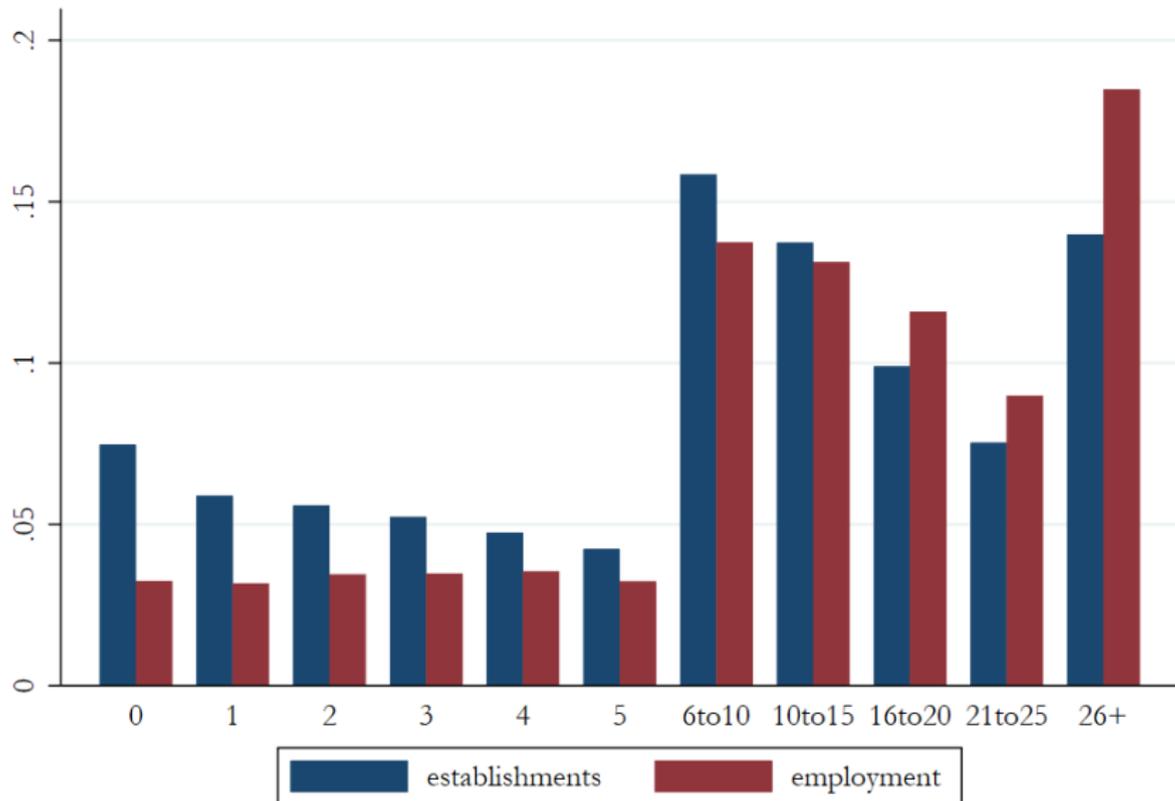
Source: Business Dynamics Statistics

Size distribution



Source: Business Dynamics Statistics 2019

Age distribution

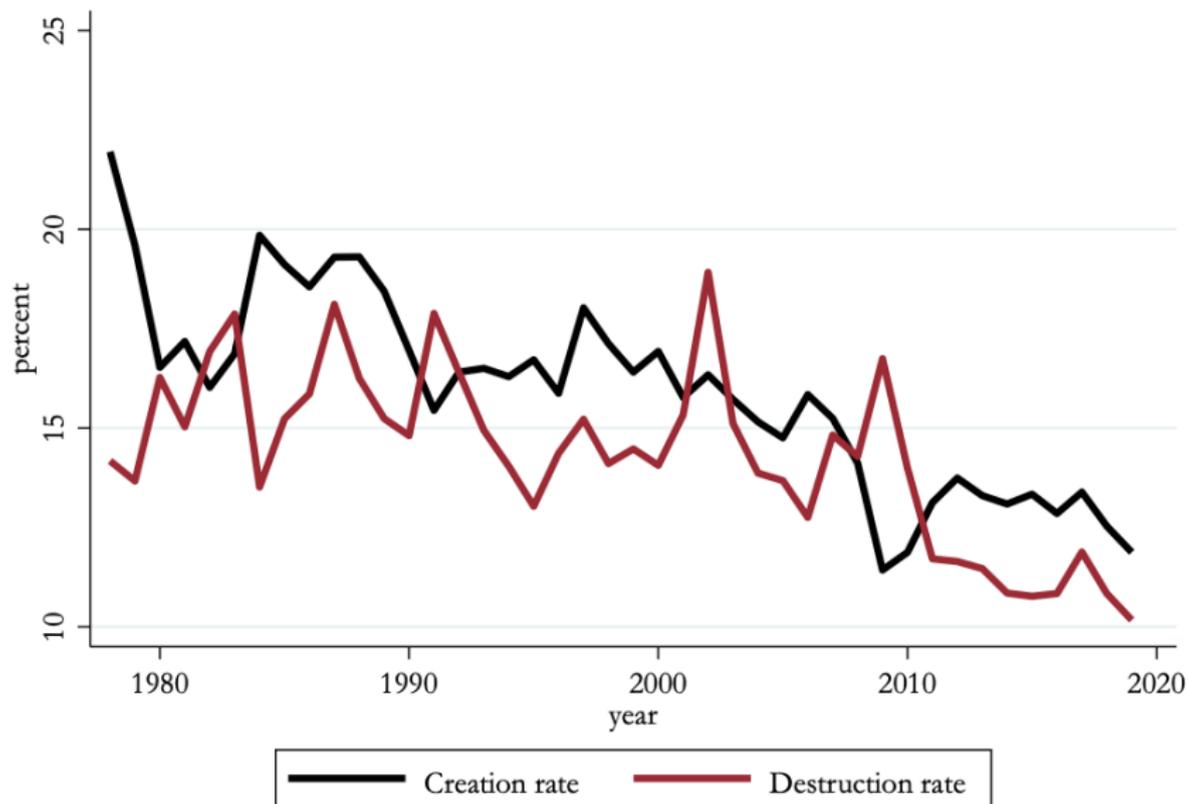


Source: Business Dynamics Statistics 2019

Taking stock

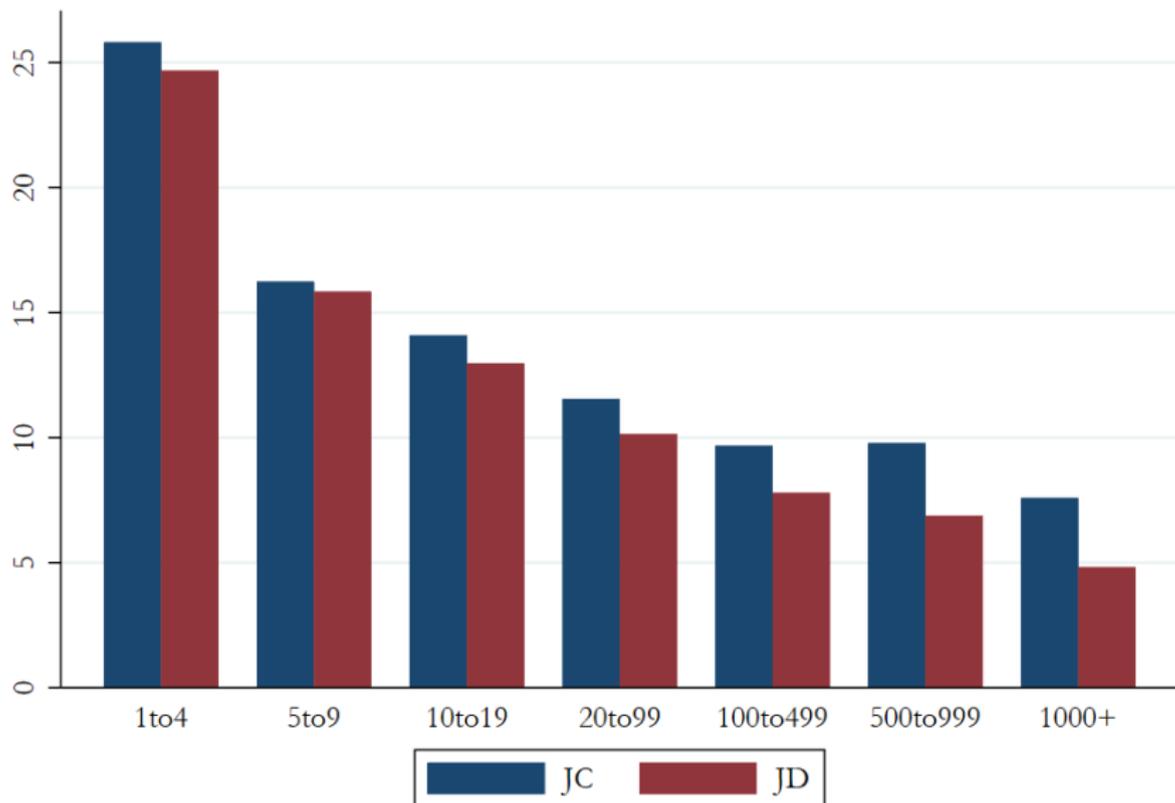
- There are about 5mil firms, comprising 7mil establishments overall
- More than 20% of employment is accounted by establishments with more than 500 employees
- On the other hand, establishments are mostly small
 - around 70% of establishments have less than 10 employees

Job creation and destruction rates



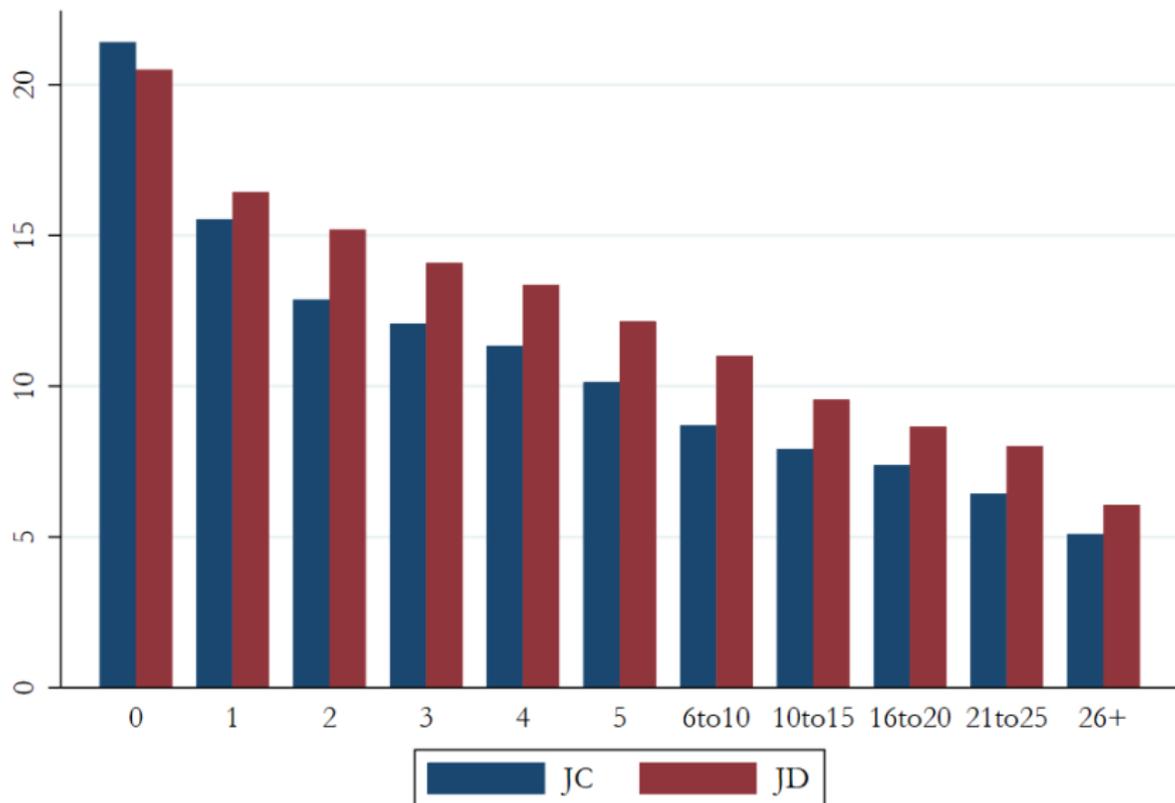
Source: Business Dynamics Statistics 2019

Job Reallocation, by size



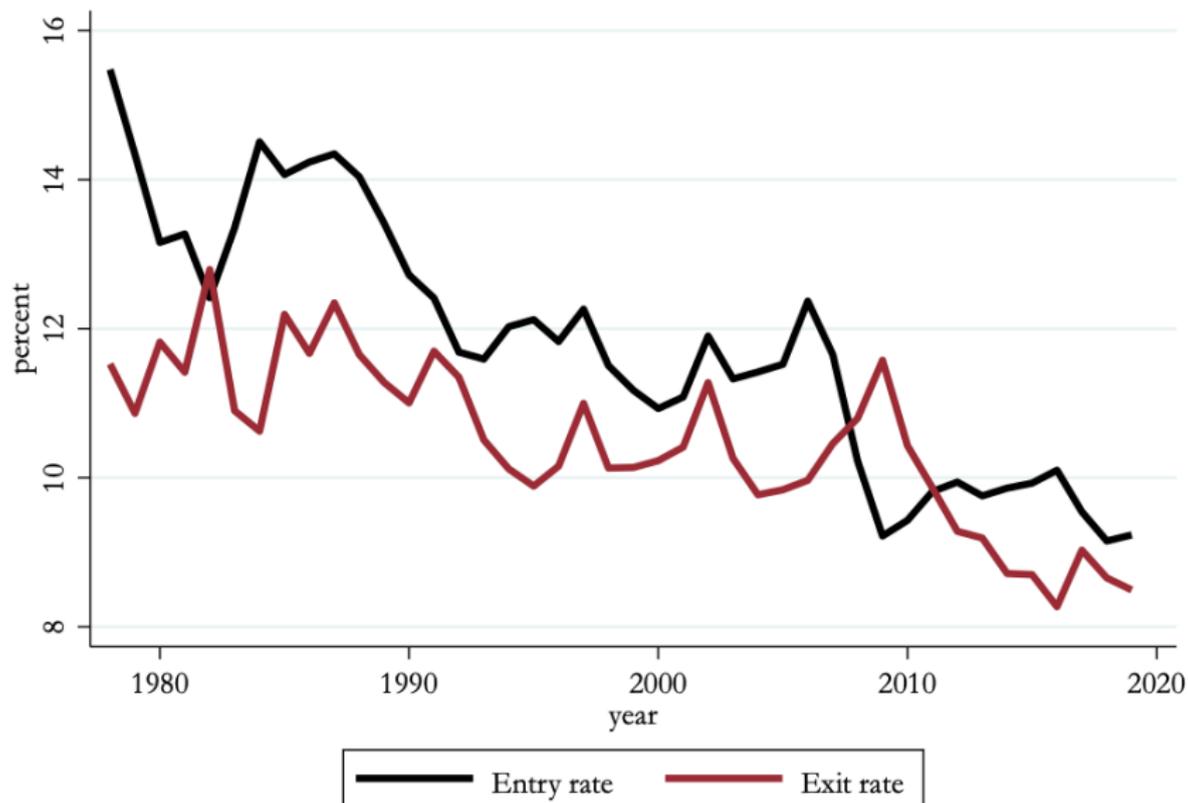
Source: Business Dynamics Statistics 2019

Job Reallocation, by age



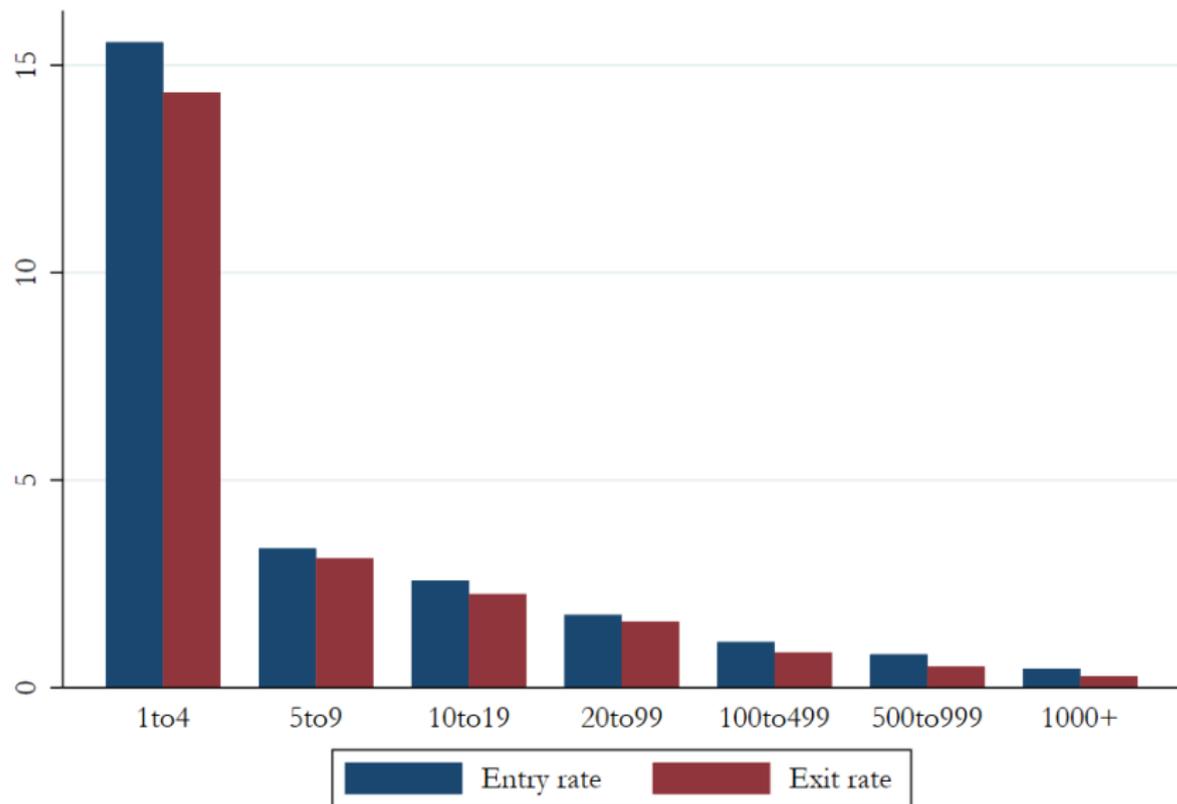
Source: Business Dynamics Statistics 2019

Firm entry and exit rates



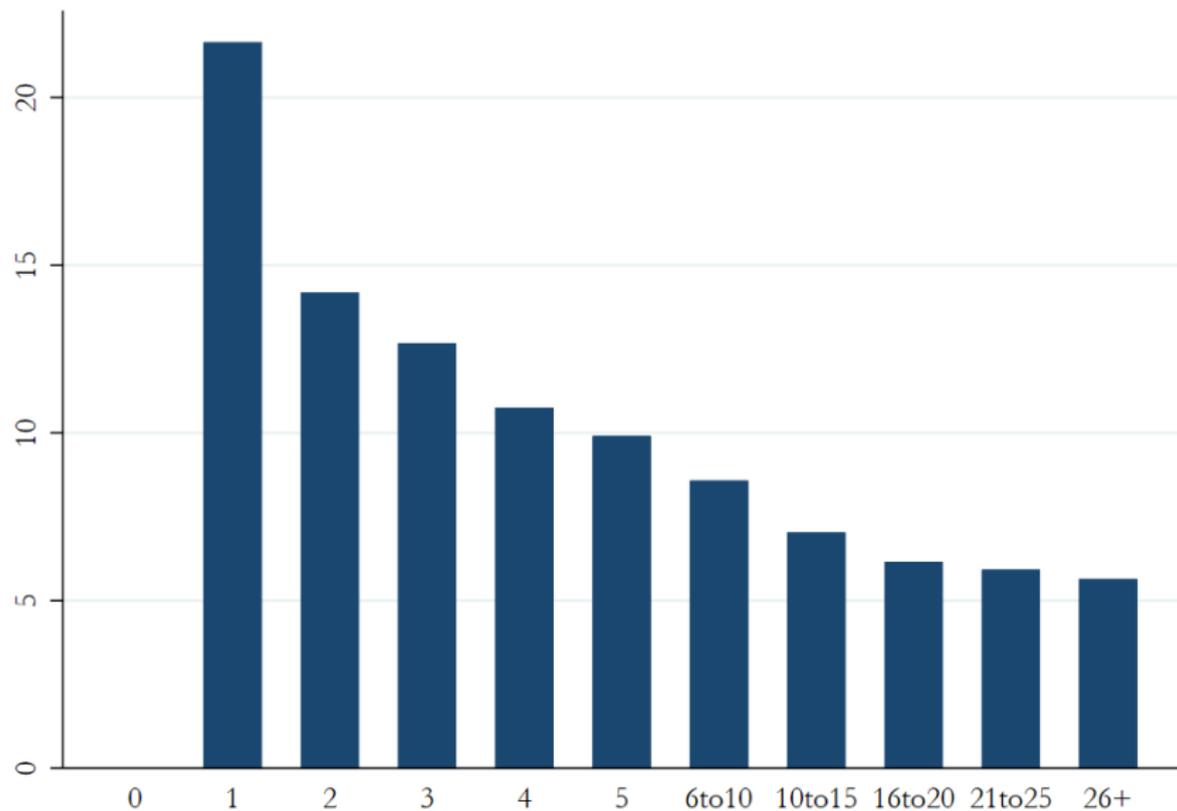
Source: Business Dynamics Statistics 2019

Turnover, by size



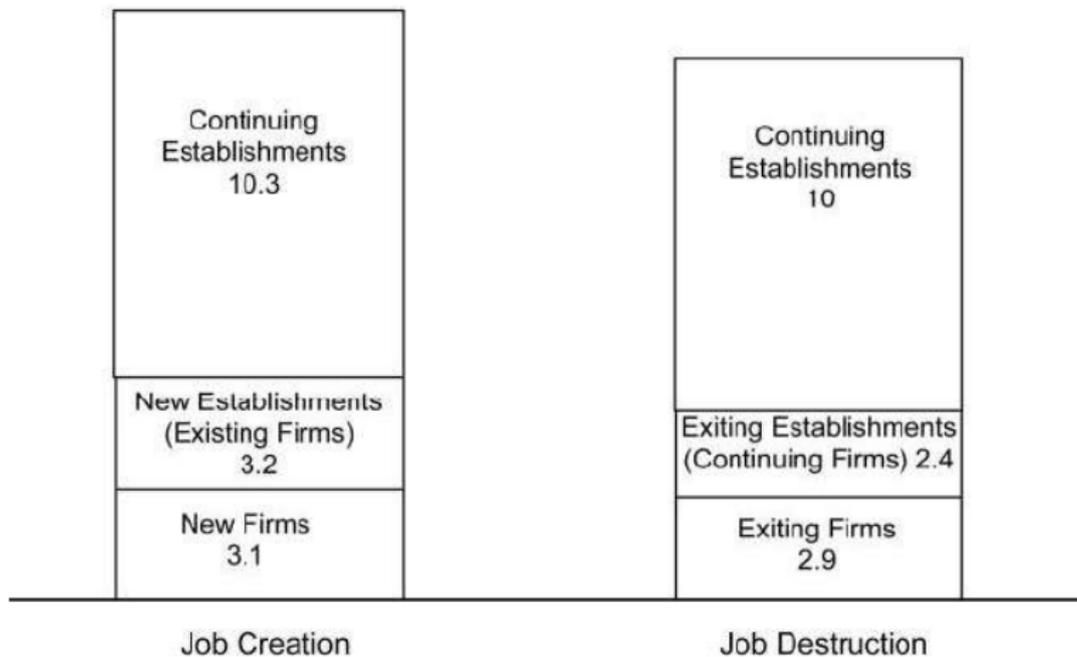
Source: Business Dynamics Statistics 2019

Exit rate, by age



Source: Business Dynamics Statistics 2019

Contribution to reallocation



Source: Haltiwanger (2012)

- Large amount of job/firm churn each year
 - 17% of all firms are either new or shut down
 - on average, 22% of all jobs are either new or destroyed
 - 40% of job creation/destruction driven by firm turnover
- Large differences heterogeneity across firms
 - young and small firms account disproportionately for job reallocation

Decomposing productivity growth

- Baily et al (1992) productivity growth decomposition:

$$\log A_t = \sum_i \theta_i \log A_{it}$$

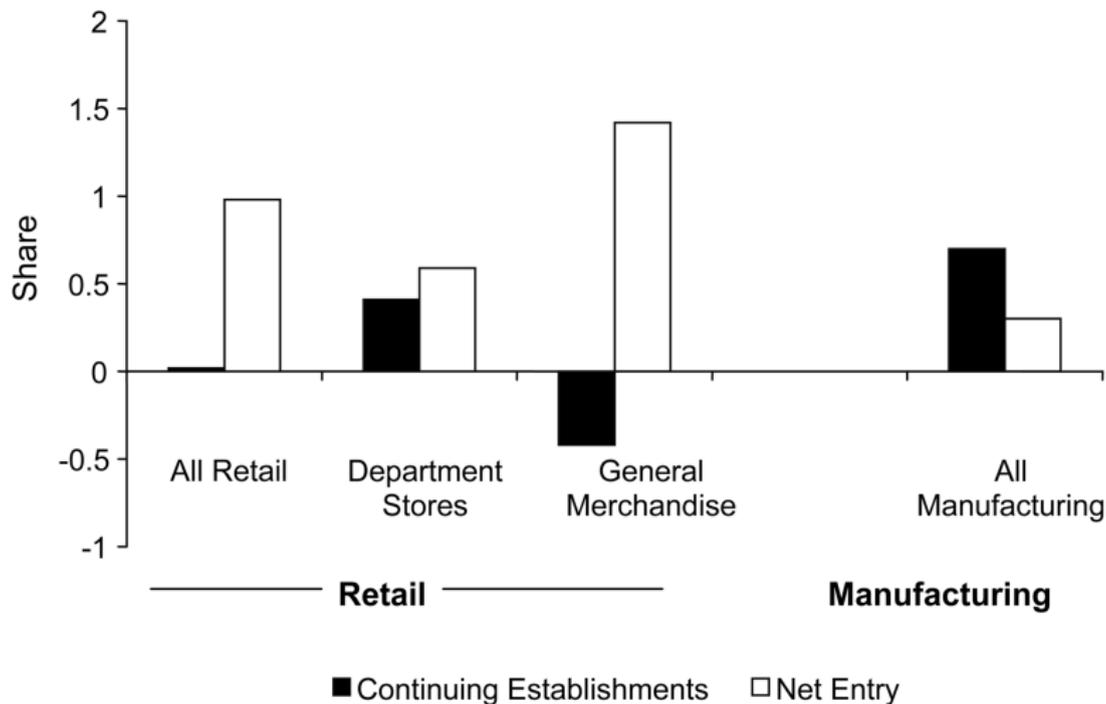
where θ_i is the output share of plant $-i$ in the market

$$\begin{aligned} \Delta \log A_t = & \underbrace{\sum_{i \in \mathcal{I}} \theta_{it-1} [\log A_{it} - \log A_{it-1}]}_{\text{within}} + \underbrace{\sum_{i \in \mathcal{I}} [\theta_{it} - \theta_{it-1}] \log A_{it}}_{\text{between}} \\ & + \underbrace{\sum_{i \in \mathcal{E}} \theta_{it} \log A_{it}}_{\text{entrants}} - \underbrace{\sum_{i \in \mathcal{X}} \theta_{it-1} \log A_{it-1}}_{\text{exiters}} \end{aligned}$$

Contribution of reallocation to firm productivity

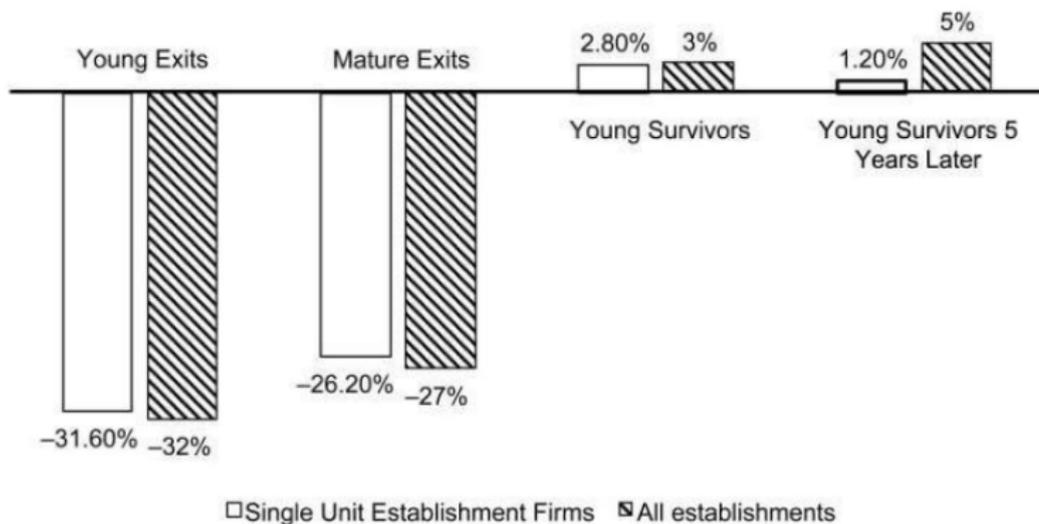
- Foster, L., J. Haltiwanger, and C. John Krizan. 2007. Aggregate Productivity Growth: Lessons from Microeconomic Evidence. University of Chicago Press.
 - US Manufacturing firms, 1978-1988
 - Reallocation of resources across heterogeneous establishments engine of aggregate productivity growth:
 - changes in market share across continuing establishments accounts for 20-30% of aggregate productivity growth
 - entry and exit dynamics accounts for more than 30% of aggregate productivity growth
 - establishments-level growth in productivity accounts for 40% of aggregate productivity growth

Contribution of net entry to productivity growth



Source: Haltiwanger (2012)

Relative productivity of young and old establishments



Source: Haltiwanger (2012)

Productivity versus Profitability

- FOSTER, Lucia; HALTIWANGER, John; SYVERSON, Chad. Reallocation, firmturnover, and efficiency: Selection on productivity or profitability?. American Economic Review, 2008, 98.1: 394-425.
 - Notation:
 - q_i : quantity produced by firm i
 - p_i : price charged by firm i
 - x_i : input expenditure by firm i
 - r_i : revenues of firm i , equal to $q_i p_i$
 - Physical Productivity: $TFPQ_i = \frac{q_i}{x_i}$
 - Revenue Productivity: $TFPR_i = \frac{p_i q_i}{x_i}$

Productivity growth

TABLE 7—DECOMPOSITION OF INDUSTRY PRODUCTIVITY GROWTH OVER FIVE-YEAR HORIZONS

Productivity measure	Total growth	Components of decomposition (BHC/FHK)					Net entry
		Within	Between	Cross	Entry	Exit	
Revenue	5.13	3.34	-0.52	1.39	0.73	0.19	0.93
Physical	5.13	3.44	-0.41	0.76	1.23	0.12	1.35
			Components of decomposition (GR)				
		Within	Between		Entry	Exit	Net entry
Revenue	5.13	4.03	0.16		0.55	0.39	0.94
Physical	5.13	3.82	-0.05		1.04	0.32	1.36

Notes: This table shows decompositions of industry-level productivity growth for three different productivity measures (shown by row) using equations (12) and (13) in text. The column labeled “Total Growth” reflects the weighted average five-year productivity growth for the industry. The remaining columns reflect the individual terms in the decomposition. Weights used in decompositions are revenue weights. Average industry revenues across the sample are used to calculate the results for the average industry. See text for details.

Source: Foster, Haltiwanger and Syverson (2008)

- Job reallocation drive productivity growth
 - reallocation due to continuing establishments and net-entry explains almost 60% of productivity growth in the manufacturing sector
- Entry-exit dynamics related to productivity growth
 - young surviving firms are more productive than old incumbents on average, and 5 years after entry
- Exit selection based on price
 - higher prices reflect both demand and cost/productivity factors
 - higher survival rates when they reflect idiosyncratic demand
 - lower survival rates if they reflect higher costs/lower productivity.

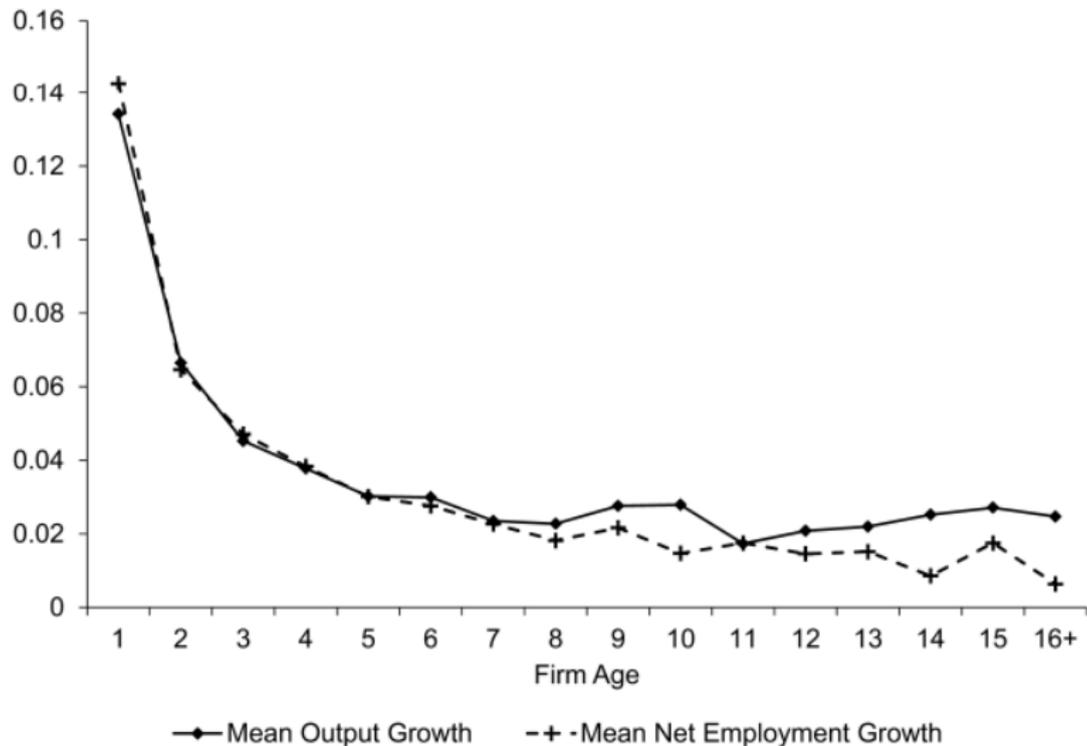
Who creates jobs? Small firms?

- Birch, David GW. 1987. "Job creation in America: How our smallest companies put the most people to work." University of Illinois at Urbana-Champaign's Academy for Entrepreneurial Leadership Historical Research Reference in Entrepreneurship
 - Old view: small firms most important for job creation
 - firms ≤ 20 employees account for 2/3 of all new jobs
 - firms ≤ 100 employees account for 4/5 of all new jobs
 - but, 98 percent of all firms have less than 100 workers
- Many countries have policies in support of small firms
- Small Business Administration in the US: "We recognize that small business is critical to our economic recovery and strength, to building America's future, and to helping the United States compete in today's global marketplace."

Who creates jobs? Young firms!

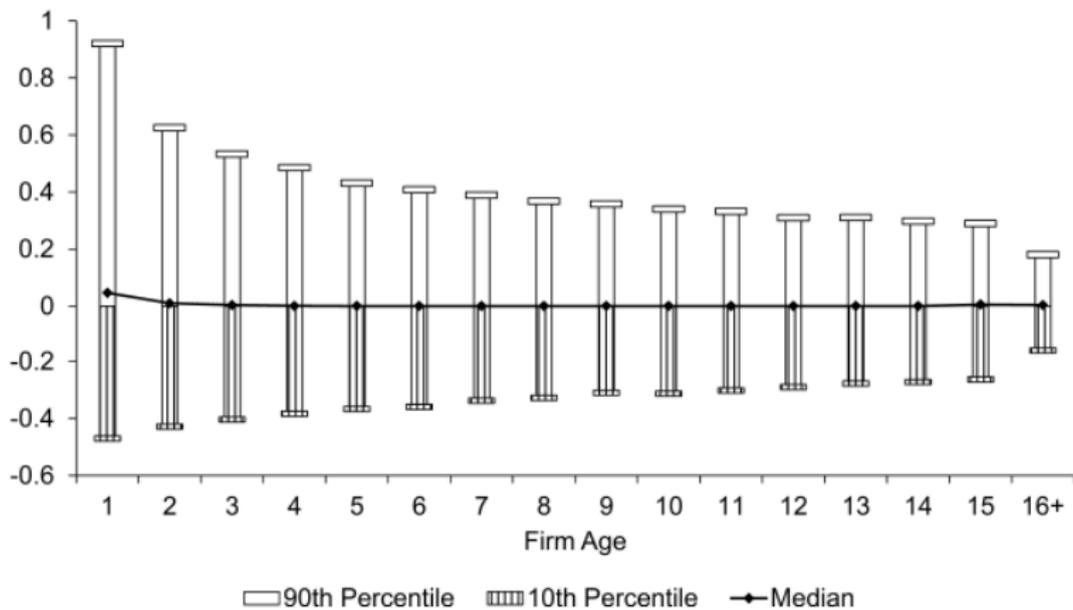
- Haltiwanger, J., R.S. Jarmin, and J. Miranda. 2013. "Who creates jobs? Small versus large versus young." Review of Economics and Statistics. Vol.95 N.2 pp.347-361.
- Haltiwanger, J. et al. 2016. "High growth young firms: Contribution to job, output and productivity growth" in "Measuring Entrepreneurial Businesses: Current Knowledge and Challenges". Ch.1 pp.11-62
 - Young firms, not small firms, are important for job creation and employment growth
 - Once controlled for firm age, there is no systematic relationship between firm size and growth

Output and employment growth



Source: Haltiwanger (2016)

Firm level employment growth distribution

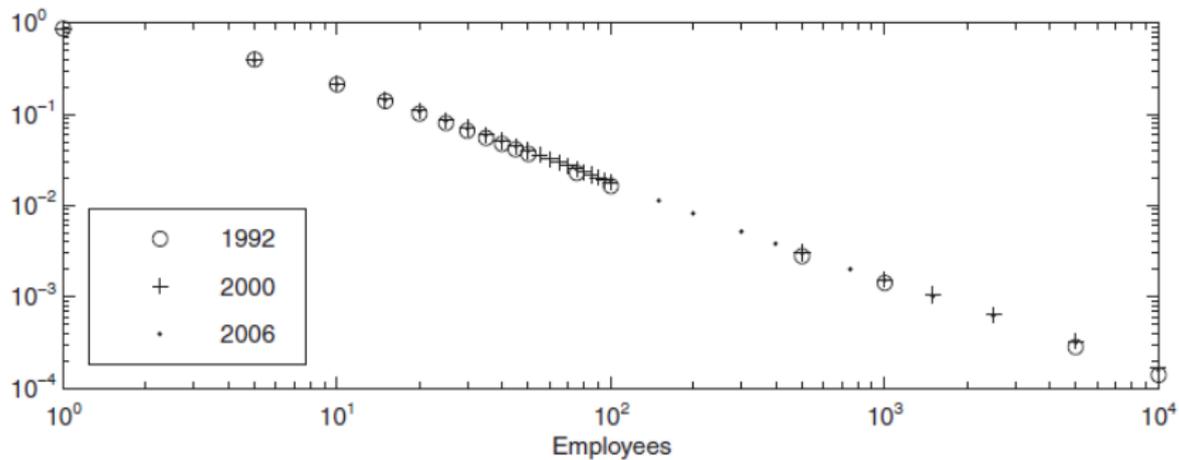


Source: Haltiwanger (2016)

- Young, not small, firm engine of growth
 - ...but median young firm does not grow at all!
- High-growth young firms, “gazelles”, account for about 10% of firms, but responsible for
 - more than 20% of firm-level gross job creation
 - 60% of aggregate job creation
 - 40% of aggregate productivity growth
 - 50% of aggregate output growth
- Sterk, V, P. Sedláček, and B. Pugsley. 2021. ”The nature of firm growth.” *American Economic Review*. Vol.111 N.2 pp.547-79.
 - are these “gazelles” different? yes!
 - differences across firms driven by ex-ante heterogeneity rather than ex-post shocks affecting incumbents

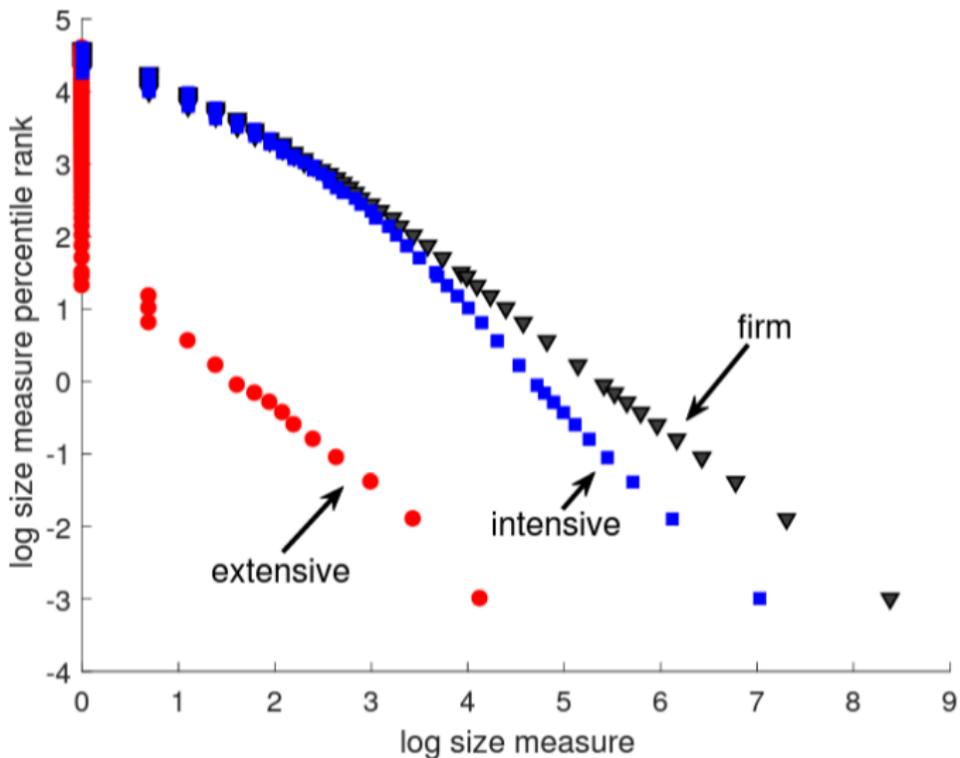
- Pareto distribution for random variable x
 - Survivor function: $\Pr[n \geq x] = x^{-\epsilon}$
 - CDF: $\Pr[n \leq x] = F(x) = 1 - x^{-\epsilon}$
 - PDF: $f(x) = \epsilon x^{-\epsilon-1}$
 - first k moments well defined if $\epsilon > k$
- Firm-size distribution well approximated by a Pareto distribution with $\epsilon \approx 1$ (thick tail)
- Zeta distribution is the discrete analogue of the Pareto
- *Zipf's law*: frequency of observation inversely proportional to rank (word-use, city-size, firm-size, ...)
 - zeta distribution with $\epsilon = 1$

Zipf's law applied to firm size



Source: Luttmer (2010)

Zipf's law applied to establishment size



Intensive= employment x establishment. Extensive=Establishment x firm.
Source: Cao et al. (2020)

A statistical model of firm growth

- *Gibrat-s law*: individual firm growth rates independent of size
 - supported by evidence, at least for enough big firms (Haltiwanger et al. 2013)
- Fixed population of firms $i = 1, 2, \dots, N$ with size n_t^i
 - average firm size: $\bar{n}_t = \frac{1}{N} \sum_i n_{it}$
 - normalized firm size: $S_t^i = n_t^i / \bar{n}_t$
- Suppose: $S_{it+1}^i = \gamma_t^i S_t^i$ with random growth: $\gamma_t^i \sim \text{iid} f(\gamma)$
- Growth $g_{it} = \frac{S_{it+1}^i}{S_{it}^i}$ is independent of current size
- Survival function: $H_t(x) = \Pr[S_t^i \geq x]$

A statistical model of firm growth

- Law of motion:

$$\begin{aligned} H_{t+1}(x) &= \Pr[S_{t+1}^i \geq x] = \Pr[\gamma_t^i S_t^i \geq x] = \Pr[S_t^i \geq \frac{x}{\gamma_t^i}] \\ &= \int_0^\infty H_t\left(\frac{x}{\gamma}\right) f(\gamma) d\gamma \end{aligned}$$

- Stationary condition: a stationary solution (if it exists) satisfies

$$\begin{aligned} H_{t+1}(x) &= H_t(x) = H(x) \quad \forall t \\ \implies H(x) &= \int_0^\infty H\left(\frac{x}{\gamma}\right) f(\gamma) d\gamma \end{aligned}$$

- Is the Pareto distribution for firm-size consistent with a stationary solution? Yes!

A statistical model of firm growth

- Guess: $H^0(x) = \Pr[S_t^i \leq x] = \kappa x^{-\epsilon}$
- Substitute it back to the stationary condition

$$1 = \int_0^{\infty} \gamma^{\epsilon} f(\gamma) d\gamma$$

which pins down the Pareto exponent ϵ as a function of f .

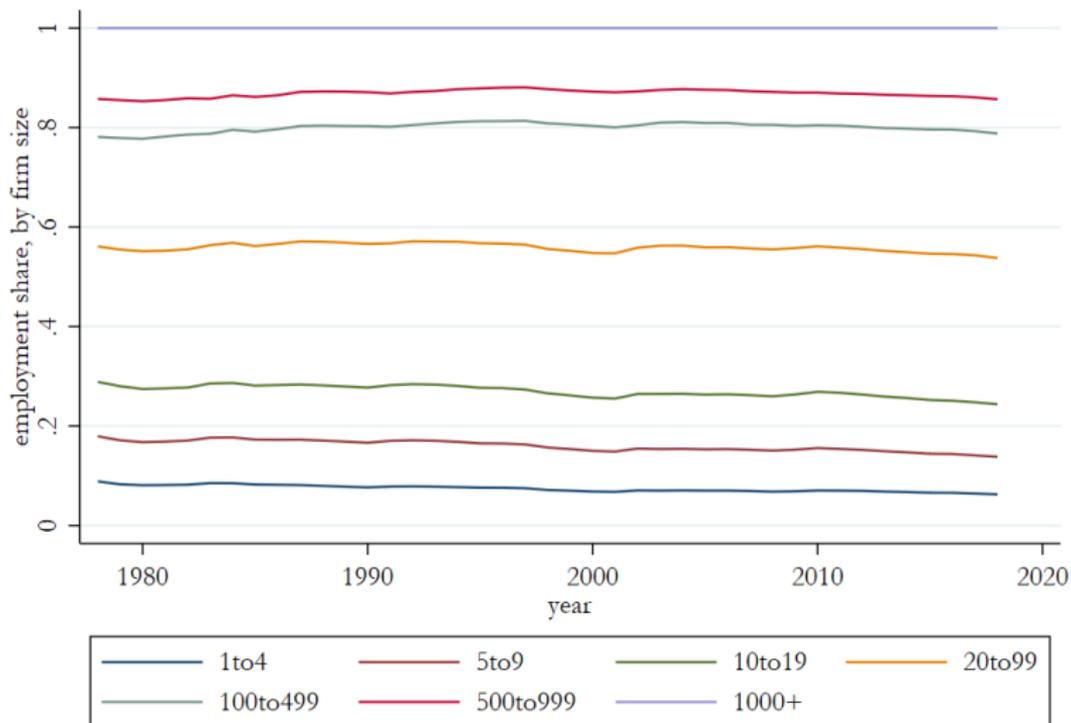
- Assume f is such that the average normalized size constant

$$\implies \mathbf{E}[\gamma_t^i] = \int_0^{\infty} \gamma f(\gamma) d\gamma = 1$$

then $\epsilon = 1$ (Zipf's law!)

- Proof for existence: Luttmer, E. 2007. "Selection, growth, and the size distribution of firms." *The Quarterly Journal of Economics* Vol.122 N.3 pp. 1103-1144.

Size distribution: is it stationary in the data ?



Source: Business Dynamics Statistics 2019

Size distribution: Is $\epsilon \approx 1$ in the data?

- Recall that the pareto distribution for firm size n implies

$$F(x) = \Pr[n \leq x] = 1 - x^{-\epsilon} \quad \text{and} \quad f(x) = \epsilon x^{-\epsilon-1}$$

- With measure M producers, total employment at plants in any size between n_1 and n_2 equal to

$$L(n_1, n_2; \epsilon) = M \int_{n_1}^{n_2} x f(x) dx = \frac{M\epsilon}{\epsilon - 1} [n_1^{1-\epsilon} - n_2^{1-\epsilon}]$$

- Implication: the share of the workforce employed at plants in the size between n_1 and n_2 equal to:

$$s(n_1, n_2; \epsilon) = \frac{L(n_1, n_2; \epsilon)}{L(1, \infty; \epsilon)} = [n_1^{1-\epsilon} - n_2^{1-\epsilon}]$$

Size distribution: Is $\epsilon \approx 1$ in the data?

- Measure $s(n_1, n_2)$ in the BDS data for selected n_1 and n_2
- Use the implied share of workforce to back out an estimate for $\hat{\epsilon}$

Size interval $[n_1, n_2]$	Employment share, $s(n_1, n_2)$	Estimate: $\hat{\epsilon}$
1-4	0.062	1.046
5-9	0.074	1.176
10-19	0.103	1.383
20+	0.761	1.092

Source: Business Dynamics Statistics

- Tybout, James. 2014. "The missing middle, revisited." Journal of Economic Perspectives. Vol.28 N.4 pp. 235-36
 - Cross country comparison in firm size distribution

- US firm size distribution characterized by
 - a stationary size distribution
 - however dynamism might change!
- Gibrat's law
 - not necessarily true for certain parts (e.g. young firms), e.g. Haltiwanger et al. (2013)
- Zipf's law
 - Pareto distribution with tail index ≈ 1 a decent approximation
 - a stationary distribution a natural result of random growth

Baseline model

Theory of production

- Standard neo-classical aggregate production function:

$$Y = AK^\alpha L^{1-\alpha} \quad \alpha \in (0, 1)$$

where Y is GDP, K is capital stock, and L is labor, and A is the total factor productivity (TFP).

- Possible interpretation: constant returns at the individual firm i with competitive markets
- Caveat: firm heterogeneity is ruled out
 - slight difference in A_i would shut down all firms except for one
 - more productive firm takes the market
- Reasonable to think that the aggregate production function is a sum of heterogeneous firms
 - productivity, size, age (life-cycle), product variety, international dimension, market power

Modelling firm heterogeneity

- Option 1: maintain the assumption of competitive markets and assume decreasing returns at firm level

$$Y = \sum_i Y_i = \sum_i A_i (K_i^\alpha L_i^{1-\alpha})^\eta \quad \alpha \in (0, 1), \eta \in (0, 1)$$

- Option 2: introduce differentiation across products i and monopolistic competition

$$Y = \left(\sum_i Y_i^\rho \right)^{\frac{1}{\rho}} = \left(\sum_i (A_i K_i^\alpha L_i^{1-\alpha})^\rho \right)^{\frac{1}{\rho}} \quad \rho \leq 1$$

- Option 3: introduce frictions in the factor markets that limit factor accumulation
 - firms expansion constrained by hiring frictions, credit constraints, ...

- Hopenhayn H. 1992. “Entry, Exit, and Firm Dynamics in Long Run Equilibrium”. *Econometrica*. Vol. 60, No. 5, pp. 1127-1150
 - Workhorse model of industry dynamics
 - Decreasing returns to scale at firm level
 - Individual dynamics: firms enter, grow, decline, and exit
 - Partial equilibrium
 - Focus on steady-state: wages and prices are constant
 - Competitive firms, no strategic interactions

- Time is discrete
- Wage is the model numeraire ($w = 1$)
- Output price p endogenous
- Endogenous measure of heterogeneous firms
 - DRS production function
 - Perfect competition in product and labor markets
 - No aggregate risk
 - Idiosyncratic risk: firm productivity follows a Markov process
 - Entry-exit dynamics
 - Fixed cost to enter
 - Fixed cost to operate each period
- Partial equilibrium: exogenous industry demand

- Firms differ in productivity z
- Firm-level output

$$f(z, n) = zn^\alpha \quad \alpha \in (0, 1)$$

- Static firm-level profits

$$\pi(z; p) = \max_{n \geq 0} pf(z, n) - n - c_o$$

where c_o denotes per-period operating costs.

- Let $n(z; p)$ denote optimal employment

$$n(z; p) = (p\alpha z)^{\frac{1}{1-\alpha}}$$

- Let $y(z; p) = f(z, n(z; p))$ denote associated output

Problem of the incumbents

- Incumbents enter the period with states z_{-1}
- Exit decision:
 - if stay, firms draw new productivity level $z \sim \Gamma(z|z_{-1})$
- Employment decision (conditional on staying)
 - choose new employment level n conditional on z
 - pay operating costs c_o and produce $y(z; p)^*$

Problem of the incumbents

- $V(z; p)$: value function for firm in states z and aggregate price p

$$V(z; p) = \pi(z; p) + \frac{1}{1+r} \tilde{v}(z; p)$$

where

$$\tilde{v}(z; p) = \max \left\{ 0, \sum_{z'} V(z'; p) \Gamma(z'|z) \right\}$$

- Solution to this problem is policy function for optimal exit: $\mathbf{1}^x(z; p)$
- An exit productivity threshold z^* exists such that:

$$\sum_{z'} V(z'; p) \Gamma(z'|z^*) = 0$$

Problem of the entrants

- Potential entrants are ex-ante identical
- New entrants $M \geq 0$ pay c_e and enter
- Draw productivity level z from a distribution $\Gamma^e(z)$
- Start producing next period
- Free entry condition:

$$v^e(p) = \frac{1}{1+r} \sum_z V(z;p) \Gamma^e(z) \leq c_e$$

with equality if $M > 0$.

Evolution of distribution

- Let $\mu_t(z; p)$ be the measure of firms over individual state z when the goods price is p at time t
- Evolution of distribution over time:

$$\mu_{t+1}(z'; p_t) = T_t(\mu_t(z; p_t), M_t, p_t)$$

where

$$T_t(\mu_t(z; p_t), M_t, p_t) = \sum_z \psi_t(z'|z; p_t) d\mu_t(z; p_t) + M_t \Gamma^e(z)$$

and

$$\psi_t(z'|z; p_t) = \Gamma(z'|z)[\mathbf{1}_t^x(z; p_t) = 0]$$

denotes the transition function from the states z to z'

Industry demand and supply

- Industry demand curve exogenous: $Y_t^d(p) = \bar{Y}$
- Industry supply curve endogenous:

$$Y_t^s(p) = \sum_z y(z; p) \mu_t(z; p)$$

- Inelastic labor supply function: $L_s^d = 1$
- Industry labor demand, endogenous,

$$L_t^d(p) = \sum_z [n(z; p) + c_o] \mu_t(z; p) + M_t c_e$$

A recursive stationary equilibrium for this economy is characterized by a measure of entrants M^* , a distribution of incumbent firms $\mu^*(z; p)$, a price p^* and a productivity threshold z^* such that the following four conditions hold:

- **Optimality:** $\forall z < z^*, \mathbf{1}^x(z) = 1$
- **Free-entry** $v^e = c_e$;
- **Goods market clearing:** $Y_t^s(p^*) = Y_t^d(p^*)$;
- **Labor market clearing:** $L_t^d = L_t^s(p^*)$;
- **Aggregate consistency:** $\mu^*(z; p^*) = T(\mu^*(z; p^*), M^*, p^*)$.

- Static version of the model
- Once-and-for-all productivity draw $z \sim \Gamma^e(z)$, which stays fixed over time, and once-and-for-all endogenous exit
- Static profit maximization problem

$$\pi(z; p) = \max_n \quad pzn^\alpha - n - c_o$$

- Solution:

- labor demand: $n(z; p) = (\alpha pz)^{\frac{1}{1-\alpha}}$
- output: $y(z; p) = \alpha^{\frac{\alpha}{1-\alpha}} (pz)^{\frac{\alpha}{1-\alpha}}$
- profits: $\pi(z; p) = [\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}](pz)^{\frac{1}{1-\alpha}} - c_o$

- Value of an incumbent:

$$v(z; p) = \max \left\{ 0, \sum_{t=0}^{\infty} \frac{\pi(z; p)}{(1+r)^t} \right\} = \max \left\{ 0, \frac{1+r}{r} \pi(z; p) \right\}$$

- Two equilibrium conditions:
 - Exit condition: $\pi(z^*; p) = 0$
 - Free entry condition:

$$c^e = \frac{1}{1+r} \sum_z v(z; p) \Gamma^e(z) = \sum_{z \geq z^*} \frac{\pi(z; p)}{r} \Gamma^e(z)$$

- Express p as a function of z^* using the exit condition:

$$\pi(z^*; p) = 0 \implies p = \frac{1}{z^*} \left(\frac{c_o}{[\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}]} \right)^{1-\alpha}$$

- Plug p into the entry condition:

$$r \frac{c^e}{c_o} = \sum_{z \geq z^*} \left(\left(\frac{z}{z^*} \right)^{\frac{1}{1-\alpha}} - 1 \right) \Gamma^e(z)$$

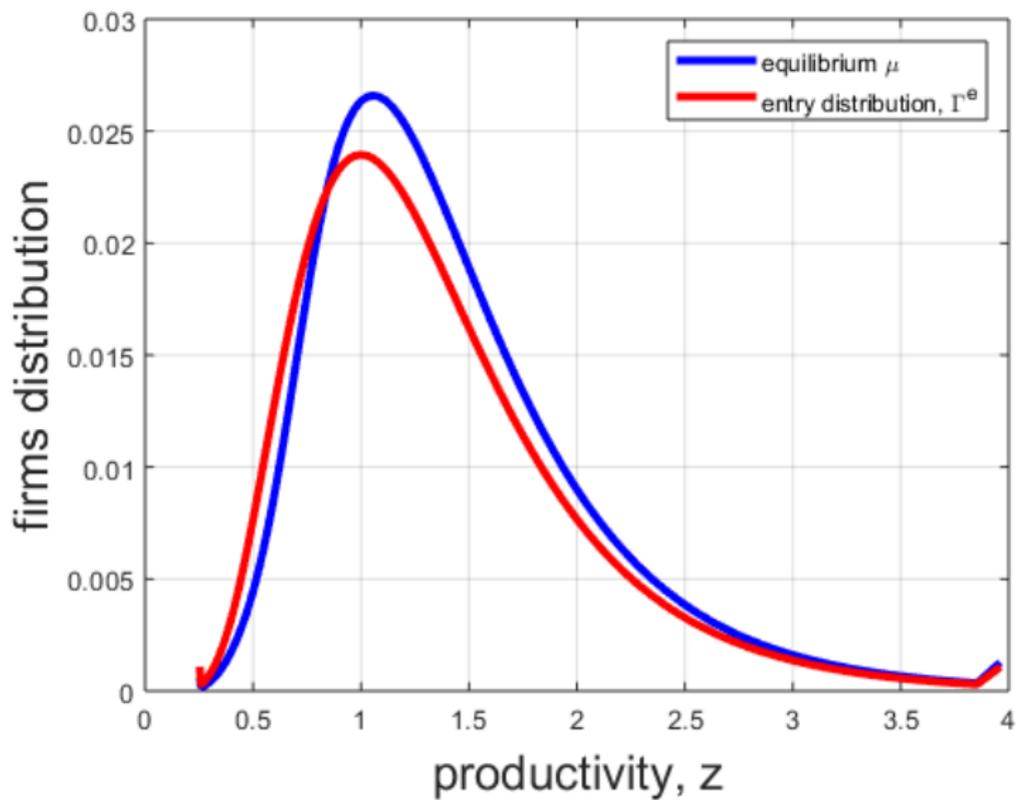
- Increase in c_e (or decrease in c_o) reduces the productivity cutoff z^* and increases p
- Larger entry barriers weaken the selection effect and allow more unproductive firms to operate

- Period model of 1 year
- Productivity follows an AR(1) process in logs:

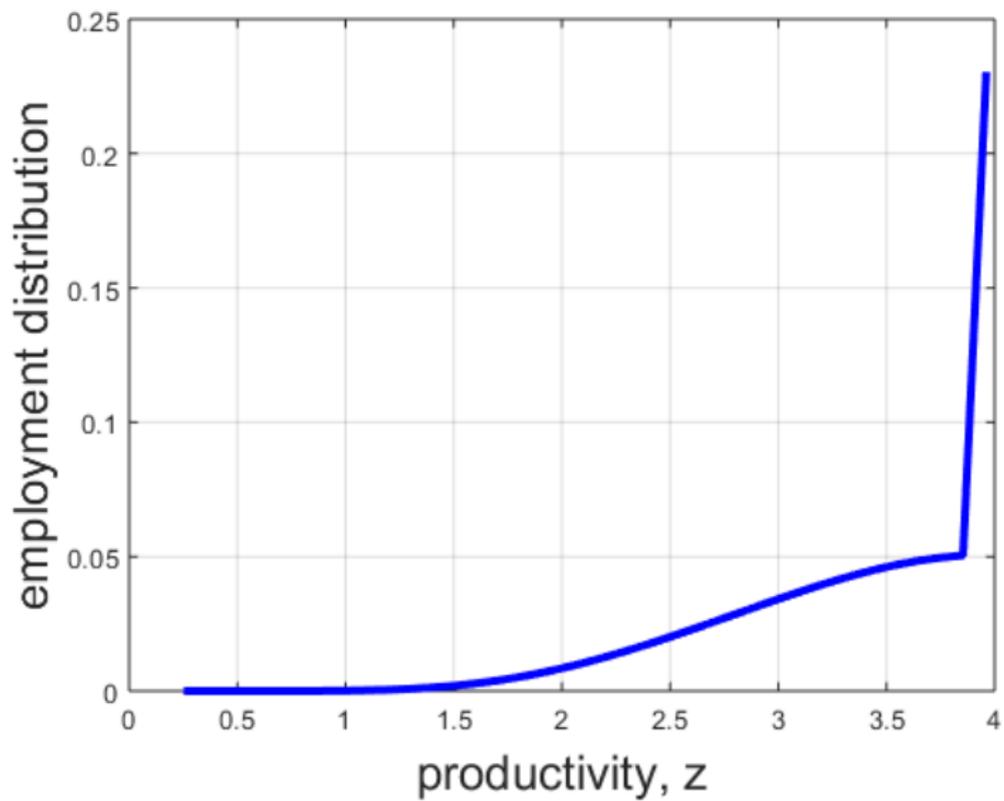
$$\log z' = \rho \log z + \epsilon' \quad \epsilon' \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

- Baseline parameters:
 - Interest rate: $r = 0.01$
 - Employment elasticity: $\alpha = 0.78$
 - Operating cost: $c_o = 10$
 - Entry: $c_e = 80$
 - Aggregate demand: $Y^d = 500$ (inelastic to aggregate price)
 - AR(1) process: $\rho = 0.995, \sigma = 0.14$

Firm distribution



Employment distribution



Comparative statics

What happens if we increase the entry cost c_e ?

Entry costs: c_e/Y	0.36	0.41	0.46	0.50
Price	1	1.162	1.247	1.316
Output	1	1.448	1.728	1.989
Profits	1	1.453	1.732	1.993
# firms	1	0.676	0.5620	0.485
Firm Size	14.429	24.295	31.081	37.799
Exit rate	47.711	44.422	43.072	42.191
Productivity threshold	1	0.741	0.652	0.598

Increase in entry costs c_e :

- decreases exit productivity threshold — **less selection!**
- decreases number of firms and exit rate
- increases prices

Two contrasting effects on firm-size

- *price/value effects*: higher c_e increases prices, which leads to higher output and employment
- *selection effects*: higher c_e reduces productivity thresholds, keeping low-productivity firms in the industry

If the density of firms near the exit point is very small, the first effect will dominate and the average value and profits increase with c_e .

Comparative statics

What happens if we increase the operating cost c_o ?

Operating costs: c_o/Y	0.04	0.08	0.13	0.20
Price	1	1.027	1.086	1.158
Output	1	1.165	1.532	1.8160
Profits	1	1.125	1.369	1.526
# firms	1	0.881	0.696	0.547
Firm size	14.429	17.270	24.010	33.337
Exit rate	47.711	51.761	57.402	61.217
Productivity threshold	1	1.408	2.159	2.791

Increase in operating costs, c_o :

- increases exit productivity threshold — **more selection!**
- decreases number of firms
- increases entry/exit rate
- increases prices

As the c_o increases, the profit function changes, pushing profits down for every state. Productivity thresholds need to increase to maintain free-entry.

Some numerical methods

- Productivity follows an AR(1) process in logs:

$$\log z' = \rho \log z + \epsilon' \quad \epsilon' \sim \mathcal{N}(0, \sigma_\epsilon^2)$$

- Discrete approximation: n -state Markov chain (Z, Γ, Γ^e)
 - number of grid points n
 - support for productivity, $Z = [z_1, z_2, \dots, z_n]$
 - transition function, $\Gamma(z_i|z_j)$, i.e.

$$\Gamma(z_i|z_j) = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \cdot & \cdot & \dots & \cdot \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}$$

with $0 \leq f_{ji} \leq 1$, $\forall i, j$, and $\sum_i f_{ij} = 1$, $\forall j$ and

$$f_{ij} = \Pr[z' = z_i | z = z_j]$$

- ergodic (stationary) distribution, $\Gamma^e(z_i) = [f_1^e, \dots, f_n^e]$

- Transitional dynamics
 - Distribution follows a deterministic difference equation
 - Start from an initial distribution f_0 at $t = 0$, where $f_{0,i} = \Pr[z_0 = z_i], \forall i = 1, \dots, n$
 - At time $t = 1$,

$$f_1 = \Gamma' f_0 \implies \begin{bmatrix} f_{1,1} \\ f_{1,2} \\ \dots \\ f_{1,n} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{21} & \dots & f_{n1} \\ f_{12} & f_{22} & \dots & f_{n2} \\ \dots & \dots & \dots & \dots \\ f_{1n} & f_{2n} & \dots & f_{nn} \end{bmatrix} \begin{bmatrix} f_{0,1} \\ f_{0,2} \\ \dots \\ f_{0,n} \end{bmatrix}$$

- At time t , $f_t = (\Gamma')f_{t-1} \implies f_t = (\Gamma')^t f_0$
- Stationary distribution:
 - Steady state of the difference equation

$$f^e = \Gamma' f^e \implies (\mathbf{I} - \Gamma')f^e = \underline{0}$$

- Existence: Γ has *only* one unit-eigenvalue

- Tauchen G. 1986. "Finite state markov-chain approximations to univariate and vector autoregressions." Economics letters. Vol.20 N.2 pp. 177-181.
- The AR(1) process generates the following conditional probabilities

$$F(z_i|z_j) = \frac{1}{\sigma_{\log z}} \phi \left(\frac{\log z_i - (1 - \rho) \log z_j}{\sigma_{\log z}} \right)$$

where ϕ is the PDF of the standard normal distribution

- If $|\rho| < 1$ then a unique, stable stationary distribution exists where

$$\sigma_{\log z}^2 = \frac{\sigma_{\epsilon}^2}{1 - \rho^2}$$

- Choose boundaries for productivity as a multiple m of the variance (usually $m = 3$)
- Set $z_n = m\sigma_{\log z}^2$
- By symmetry of the normal distribution, set $z_1 = -z_n$
- Let z_2, z_3, \dots, z_{n-1} be located in a equispaced manner over the interval $[z_1, z_n]$
- Compute probability that z' falls within a distance d from each point in the grid, i.e.

$$\begin{aligned}
 f_{ij} &= \Pr(z_i - d/2 < z' < z_i + d/2) \\
 &= \Pr(z_i - d/2 < \rho z_j + \epsilon < z_i + d/2) \\
 &= \Pr(z_i - \rho z_j - d/2 < \epsilon < z_i - \rho z_j + d/2) \\
 &= F(z_i - \rho z_j + d/2 | z_j) - F(z_i - \rho z_j - d/2 | z_j)
 \end{aligned}$$

- Suppose productivity follows n -state Markov chain on z_j with transition probabilities f_{ij}
- Given price p , the value function for the incumbent firms is a n -vector with elements equal to

$$\begin{bmatrix} v_1(p) \\ v_2(p) \\ \dots \\ v_n(p) \end{bmatrix} = \begin{bmatrix} \pi_1(p) \\ \pi_2(p) \\ \dots \\ \pi_n(p) \end{bmatrix} + \frac{1}{1+r} \begin{bmatrix} \max \{0, \sum_{i=1}^n v_i(p) f_{i1}\} \\ \max \{0, \sum_{i=1}^n v_i(p) f_{i1}\} \\ \dots \\ \max \{0, \sum_{i=1}^n v_i(p) f_{in}\} \end{bmatrix}$$

- Stacking the values into a vector v , this is a mapping of the form

$$v(p) = T(v, p)$$

- A solution $v^*(p)$ is fixed point of T , i.e. a function such that:

$$v^*(p) = T(v^*(p), p)$$

- Definition: A function T on a metric space X with a distance function $d : X \rightarrow \mathcal{R}$ is a *contraction mapping* if there exists non-negative real number $0 \leq k < 1$ such that, for some $x, y \in X$:

$$d(T(x), T(y)) < kd(x, y)$$

- *Blackwell's theorem*: A function $T : X \rightarrow \mathcal{R}$ is a contraction mapping if the following two sufficient conditions are satisfied:
 - monotonicity: given a real number, $x \in \mathcal{R}$, and two functions $f, g \in M$, with $h(x) \leq g(x)$,

$$T(h(x)) \leq T(g(x))$$

- discounting: there exists some $\beta \in (0, 1)$ such that for any function $h \in M$

$$T((h + a)(x)) \leq T(h(x)) + \beta a \quad a \geq 0, x \in X.$$

- *Contraction mapping theorem:* If a function $T : X \rightarrow \mathcal{R}$ is a contraction mapping then there exists a function v^* such that $v^* = T(v^*)$.

Moreover, the solution is unique and can be found using an iterative method:

- Set a tolerance level, $\varepsilon = e^{-06}$.
- Fix p . Guess some initial $v^0 = [v_1^0, v_2^0, \dots, v_n^0]$.
- Compute $v^1(p) = T(v^0, p)$. Use $v^1(p)$ as a new guess.
- After k iterations: $v^{k+1}(p) = T(v^k, p)$
- Stop if $|v^{k+1}(p) - v^k(p)| < \varepsilon$.

- Given $v^*(p)$, the productivity threshold for exit z^* can be obtained as:

$$z^* = z_{i^*} \quad i^* \equiv \min_i \left[\sum_{j=1}^n v_j^*(p) f_{ij} \geq 0 \right]$$

- All firms with $z_i < z_{i^*}$ exit, all firms with $z_i \geq z_{i^*}$ continue
- A solution is a vector $\mathbf{1}_i^x(p) = \begin{cases} 1 & \text{if } z_i < z_{i^*} \\ 0 & \text{otherwise} \end{cases}$

- Given $v^*(p)$ that solves incumbent's problem, value of entry is

$$v^*(p) = \frac{1}{1+r} \sum_{j=1}^n v_j^*(p) f_j^e$$

- Equilibrium price p^* is determined by the free-entry condition

$$\frac{1}{1+r} \sum_{j=1}^n v_j^*(p^*) f_j^e = c_e$$

- Notice that $v^e(0) < 0$ and $v^e(p)$ is monotonically increasing in p , hence any interior solution p^* can be found using *bisection*
 - if $v^e(p) > c_e$ then reduce price to discourage entry
 - if $v^e(p) < c_e$ then increase price to encourage entry

- **Step 0:** discretize productivity state-space
- **Step 1:** guess a price p_0
- **Step 2:** solve for the value of the incumbent, $v(z; p^0)$. A solution to this problem implies an optimal exit rule, $\mathbf{1}^x(z; p^0)$
- **Step 3:** compute the value of entry, v^e and check if free-entry condition is satisfied:
 - if no, make a new guess p_1 and go back to step 2 till convergence
 - if yes, store $p^* = p_0$

- **Step 4:** given p^* , solve for the stationary distribution of incumbents $\mu(z; p^*)$ and measure of entrants M^e
 - guess measure of entrants M_0^e
 - calculate the stationary distribution $\mu(z; p^*, M_0^e)$
 - given $\mu(z; p^*, M_0^e)$, compute total industry supply and check the if market clearing condition is satisfied:
 - if no, make a new guess M_1^e and go back to step 4 till convergence
 - if yes, store $M^{e*} = M^e$

- Steps 4 can be speeded up exploiting the property of linear homogeneity of the function $\mu(\cdot)$ with respect to M
- Let $\mu_{i,t} = \mu_t(z_i)$ be the share of firms with productivity z_i
- Vector μ_t evolves according to difference equation

$$\begin{bmatrix} \mu_{1,t}(p) \\ \mu_{2,t}(p) \\ \dots \\ \mu_{n,t}(p) \end{bmatrix} = \begin{bmatrix} f_{11}(1 - \mathbf{1}_t^x(z_1; p)) & \dots & f_{n1}(1 - \mathbf{1}_t^x(z_1; p)) \\ f_{12}(1 - \mathbf{1}_t^x(z_2; p)) & \dots & f_{n2}(1 - \mathbf{1}_t^x(z_2; p)) \\ \dots & \dots & \dots \\ f_{1n}(1 - \mathbf{1}_t^x(z_n; p)) & \dots & f_{nn}(1 - \mathbf{1}_t^x(z_n; p)) \end{bmatrix} \begin{bmatrix} \mu_{1,t-1}(p) \\ \mu_{2,t-1}(p) \\ \dots \\ \mu_{n,t-1}(p) \end{bmatrix} + M \begin{bmatrix} f_1^e \\ f_2^e \\ \dots \\ f_n^e \end{bmatrix}$$

- Impose stationarity: $\mu_t(p) = \mu_{t-1}(p) = \mu(p)$

- Re-write in matrix notation the function μ , i.e.

$$\boldsymbol{\mu}(p, M) = \boldsymbol{\Psi}(p)\boldsymbol{\mu}(p) + M\boldsymbol{\Gamma}^e$$

- Solution of $\boldsymbol{\mu}$ defined up to a value for M :

$$\boldsymbol{\mu}(p, M) = M(\mathbf{I} - \boldsymbol{\Psi}(p))^{-1}\boldsymbol{\Gamma}^e$$

- Homogeneity of first degree:

$$\boldsymbol{\mu}(p, M) = M\boldsymbol{\mu}(p, 1)$$

- Compute equilibrium M using market clearing condition
 - Supply is endogenous: $\sum_{i=1}^n y(z_i; p^*) \mu_i(p^*, M^*)$
 - Demand is exogenous (partial equilibrium): \bar{Y}
 - Equating demand and supply:

$$\sum_{i=1}^n y(z_i; p^*) \mu_i(p^*, M^*) = \bar{Y}$$

$$\sum_{i=1}^n y(z_i; p^*) M^* \mu_i(p^*, 1) = \bar{Y}$$

which implies

$$M^* = \frac{\bar{Y}}{\sum_{i=1}^n y(z_i; p^*) \mu_i(p^*, 1)}$$

Computation

- What if corner solution, i.e. $M = 0$? No entry/exit dynamics
- The stationary distribution of firms just given by stationary distribution of Markov chain, i.e.

$$\mu(z; p) = \sum_{z'} \Gamma(z'|z) = \mu(z)$$

- Stationary distribution independent of p - no need to use free entry condition!
- Solve for p^* using the market clearing condition, i.e.

$$Y^s(p^*) = \bar{Y}$$