Chapter 3

Discrete random variables

Exercise 1 Expected value and variance of X are equal to:

$$\mathrm{E}[X] = 0(0.25) + 1(0.50) + 2(0.25) = 1$$
$$\mathrm{VAR}[X] = \big[(0-1)^2(0.25)\big] + \big[(1-1)^2(0.5)\big] + \big[(2-1)^2(0.5)\big] = 0.25 + 0.25 = 0.5$$

Exercise 2 The marginal probability distributions for X is equal to:

$$P(X = 0) = 0.50 \quad P(X = 1) = 0.50$$

The marginal probability distributions for Y is equal to:

$$P(Y = 1) = 0.55 \quad P(Y = 2) = 0.45$$

The expected value of X and Y are equal to:

$$\mathrm{E}[X] = 0(0.50) + 1(0.50) = 0.50$$
$$\mathrm{E}[Y] = 1(0.55) + 2(0.45) = 1.45$$

The variance of X and Y are equal to:

$$\mathrm{VAR}[X] = (0 - 0.50)^2(0.50) + (1 - 0.5)^2(0.50) = 0.25$$
$$\mathrm{VAR}[Y] = (1 - 1.45)^2(0.55) + (2 - 1.45)^2(0.45) = 0.2475$$

The probability of having one exam conditional on eating two snacks is:

$$P(X = 1|Y = 2) = P(X = 1, Y = 2)/P(Y = 2) = \frac{0.25}{0.45} \approx 0.56$$

The covariance of X and Y is equal to

$$\begin{aligned}\mathrm{COV}[X,Y] = (0-0.50)(1-1.45)0.3 + (1-0.50)(1-1.45)0.25 + \\ +(0-0.50)(2-1.45)0.20 + (1-0.50)(2-1.45)0.25 = 0.025\end{aligned}$$

Exercise 3 Notice that:

- Number of trials: $n = 6$
- Number of successes: $x = 2$
- Probability of success: $p = 0.6$

The Binomial probability formula is:

$$P(X = x) = C_x^n p^x (1-p)^(1-x)$$

where

$$C_x^n = \frac{n!}{x!(n-x)!}$$

Since $C_x^n = \frac{6!}{2!4!} = 15$, then

$$P(X = 2) = 15(0.6^2 0.4^4) = 15(0.009) = 0.138$$

Exercise 4 Notice that

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

Since:

$$\begin{aligned}P(X = 0) = C_0^{12}(0.1^0)(0.9^{12}) = 0.282 \\ P(X = 1) = C_1^{12}(0.1^1)(0.9^{11}) = 0.377 \\ P(X = 2) = C_2^{12}(0.1^2)(0.9^{10}) = 0.230\end{aligned}$$

Then:

$$P(X \leq 2) = 0.282 + 0.377 + 0.230 = 0.889$$

The probability of a shipment containing 3 or more faulty phones is $1 - P(X \leq 2) = 1 - 0.889 = 0.111$. Let Y be the number of shipments containing 3 or more faulty phones. Y can be modelled with a binomial distribution with $n = 6$ and $p = 0.111$. The probability that at least one shipment will contain 3 or more faulty phones is:

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - C_0^6(0.111^0)(0.889^6) = 1 - 0.494 = 0.506$$

Finally, notice that:

- Expected value: $\mu = np = 6(0.111) = 0.666$
- Variance: $\sigma^2 = np(1-p) = 6(0.111)(1-0.111) = 0.592$
- Standard deviation: $\sigma = \sqrt(0.592) = 0.769$

The probability that Y exceeds its mean by more than 2 standard deviations is

$$P(Y \geq \mu + 2\sigma) = P(Y \geq 0.666 + 2(0.769)) = P(Y > 2.205)$$

As Y is discrete, this is equal to $P(Y > 3) = 1 - [P(Y = 0) + P(Y = 1) + P(Y = 2)]$. Since:

$$P(Y = 0) = C_0^6(0.111^0)(0.889^6) = 0.494$$
$$P(Y = 1) = C_1^6(0.111^1)(0.889^5) = 0.370$$
$$P(Y = 2) = C_2^6(0.111^2)(0.889^4) = 0.115$$

then $P(Y > 3) = 1 - 0.494 - 0.370 - 0.115 = 0.021$.