Mathematics - Brush-up
Problem Set 1

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1 Metric Spaces

Exercise 1
For each pair $(x, y) \in \mathbb{R} \times \mathbb{R}$, let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by:

1. $d(x, y) = \frac{|x-y|}{1+|x-y|}$
2. $d(x, y) = |x - 2y|$
3. $d(x, y) = (x - y)^2$
4. $d(x, y) = \sqrt{|x - y|}$

Are $d(x, y)$ a metric? Prove it.

Exercise 2
Show that for $p > 1$ the function

$$d(x, y) : \mathbb{R}^n \to \mathbb{R}_+ \text{ s.t. } d(x, y) = \left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}}$$

defines $(d_p, \mathbb{R}^n)$ as a metric space. *Hint: To prove triangle inequality use the property called Minkowsky inequality 1:*

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}}$$

Exercise 3
Consider the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by the **British Rail** distance, $d(x, y) = ||x|| + ||y||$ if $x \neq y$, 0 otherwise, where the function $|| \cdot ||$ is, in turn, 1) the taxicab norm and 2) the euclidean norm. Check whether d defines a metric function for each type of norm.

Exercise 4
Consider two metric spaces (d, X) and (d', X). Define $d_{\max} = \max(d, d')$ and $d_{\min} = \min(d, d')$. Is $(d_{\max}, X)$ a metric space? Is $(d_{\min}, X)$ a metric space? Prove it.

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2 Sequences

Exercise 5
Assume $x \in (0,1)$. Prove that the sequence $(1+x, 1+x+x^2, 1+x+x^2+x^3, ...)$ converges to $\bar{x} = \frac{1}{1-x}$.

Exercise 6
Show that the sum of two convergent sequences in a generic normed space is a convergent sequence.

Exercise 7
Prove that, for all non-increasing function mapping naturals to positive reals, $f : \mathbb{N} \to \mathbb{R}_+$ (that is $f(i) \leq f(j)$, $\forall i \geq j$) the sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = \frac{f(n)}{n}$ is a Cauchy sequence.

Exercise 8:
Prove that every discrete metric space (i.e. a space endowed with a discrete metric) is complete.

3 Open and Closed Sets

Exercise 9:
Given a space X, two metrics d and $d^{'}$ are said to be **strongly equivalent** in X if $\exists \alpha, \beta \in \mathbb{R}_+$ s.t. $\forall x, y \in X$, $\alpha d(x,y) \leq d^{'}(x,y) \leq \beta d(x,y)$. Strong equivalence implies topological equivalence; that is, if two metrics are strongly equivalent in X, then they generate the same topology on X. Using this notion, show that statements 1 and 2 are equivalent, while statement 2 and 3 are **not** equivalent:

1. Under $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, $\forall x, y \in S \subseteq \mathbb{R}^n$, S is an open set in $\mathbb{R}^n$
2. Under $d_1(x,y) = \max_i |x_i - y_i|$, $\forall x, y \in S \subseteq \mathbb{R}^n$, S is an open set in $\mathbb{R}^n$
3. Under the discrete metric, d_d, S is an open set in $\mathbb{R}^n$

Exercise 10:
Is the intersection of an arbitrary family of open intervals open? Why? Prove it.

Exercise 11:
A set $A \subset X$ is said to be **closed** if, given $x \in X$ s.t. $d(x,A) = \inf_{a\in A}(d(x,a)) = 0$ then $x \in A$. Prove that this definition is equivalent to the notion of closed set that says: a set A is closed if and only if it coincides with its closure.

4 Boundedness and Compactness

Exercise 12:
Prove that a finite union of bounded sets is a bounded set.

Continuity

Exercise 13:
Let $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ be two continuous functions. Is it $f : \mathbb{R} \to \mathbb{R}$ given by

1. $g(x) = \min\{f_1(x); f_2(x)\}$
2. $h(x) = \max\{f_1(x); f_2(x)\}$

continuous?

Exercise 14:
Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that if $f(x) > 0$ for an element $x \in X$ then there exists an open subset O in X such that $f(y) > 0$ for all $y \in O$.

Exercise 15:
Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then for any $\alpha \in \mathbb{R}$, the set $S_\alpha : \{x \in \mathbb{R} : f(x) \geq \alpha\}$ is closed.

Exercise 16:
Is the function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ continuous in $\mathbb{R}_+$? What about in $\mathbb{R}_{++}$? Is it uniformly continuous in $\mathbb{R}_{++}$? What about in $X = [\alpha, \infty)$? Prove it.

5 Convexity

Exercise 17:
Prove that the intersection of any collection of convex set is a convex set.

Exercise 18:
Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function and $g : \mathbb{R} \to \mathbb{R}$ a quasi-concave function. Is $h = f + g$ necessarily quasi-concave? Prove it.

Exercise 19:
The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are concave. Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $h = fg$ necessarily quasi-concave?