Firms and the labour market:

Theories and structural estimation

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UoN, PhD expert class

Outline

1. Lecture 1: Firm dynamics with frictional labor market
2. Lecture 2: Firm growth with on-the-job search
3. Lecture 3: Structural estimation and identification

1. Lecture 1: Firm dynamics with frictional labor market
2. Lecture 2: Firm growth with on-the-job search
3. Lecture 3: Structural estimation and identification

- Elsby M. and Michaels R. 2013. "Marginal Jobs, Heterogeneous Firms, and Unemployment Flows." American Economic Journal: Macro, Vol. 5, No. 1, pp. 1-48
  - Multi-worker heterogeneous firms $\implies$ well-defined firm-size
  - Idiosyncratic productivity shocks $\implies$ endogenous job destruction
  - Search frictions $\implies$ unemployment
  - Wage bargaining $\implies$ wage dispersion

- Entry/exit dynamics
- Firing costs
    - Bertola G. and Caballero R. 1994. “Cross-Sectional Efficiency and Labour Hoarding in a Matching Model of Unemployment”. Review of Economic Studies, Vol. 61, No. 3, pp. 435-456
- Heterogeneous agents:
    - Cahuc P., Marque F. and Wasmer E. 2008. “A Theory of Wages and Labor Demand with Intra-Firm Bargaining and Matching Frictions.” International Economic Review, Vol. 49, No. 3, pp. 943-972
- Aggregate shocks

The baseline model

- The labor market is in a perpetual state of flux
  - Workers move across labor market states
  - Firms grow/shrink via job creation/destruction
  - Worker and job flows are large and correlated
- Goal: Synthesis of these ingredients
- How would changes in labor demand, induced by the changes in firm dynamics, affect unemployment?

- Time is discrete
- Exogenous measure of potential workers L
  - infinitely lived
  - risk neutral
  - homogeneous
  - employed/unemployed
- Unitary measure of producers
  - heterogeneous in productivity
    - fixed productivity level, s
    - time-varying productivity level, z
  - no entry/exit dynamics
  - job creation/destruction
  - search frictions and wage bargaining
  - no aggregate shocks

- Decreasing return to scale firm-level production function

$$y = szF(n)$$

where $F'(n) > 0$ and $F''(n) < 0$

  - the marginal product of labor declines with firm employment, and generates a downward-sloped demand for labor at the firm level

- Time invariant productivity $s \sim \mathcal{H}(s)$
- Idiosyncratic productivity z assumed to follow a markov chain $z' \sim \Gamma(z'|z)$

- Labor market is subject to search and matching functions
    - Workers need to search in order to find jobs
    - Firms need to post vacancies in order to attract workers
- Only unemployed workers can search - no on-the-job search
- The matching process between vacancies V and unemployed workers U is governed by a CRS matching function $m(U, V)$:
    - increasing in both arguments

$$\frac{\partial m(U,V)}{\partial U} > 0 \quad \frac{\partial m(U,V)}{\partial V} > 0$$

    - concave in both arguments

$$\frac{\partial \partial m(U,V)}{\partial U^2} < 0 \quad \frac{\partial \partial m(U,V)}{\partial V^2} < 0$$

    - homogeneous of degree one in both arguments

- Cobb-Douglass function:

$$m(U,V) = m_0 U^{\alpha} V^{1-\alpha} \quad m_0 > 0, \alpha \in [0,1]$$

where

  - m_0 stands for matching efficiency
  - α stands for the elasticity of the matching function with respect to unemployment

- Other functional form (Den Haan et al., 2001)

$$m(U,V) = \frac{UV}{(U^{\eta} + V^{\eta})^{\frac{1}{\eta}}} \quad \eta > 0$$

where the elasticity of matching function equal to:

$$\frac{\partial m(U,V)}{\partial U} \frac{U}{m(U,V)} = \left(1 + \left(\frac{V}{U}\right)^{\eta}\right)^{-\frac{1}{\eta}}$$

- Labor market tightness: $\theta = V/U$
- Search frictions in the labor market limit the rate at which unemployed workers and hiring firms can meet:
  - Job finding probability for workers:

$$\phi_w = \frac{m(U,V)}{U} = m\left(1, \frac{V}{U}\right) = m_0\theta^{1-\alpha}$$

  - Vacancies posted by firms are filled with probability:

$$\phi_f = \frac{m(U,V)}{V} = m\left(\frac{U}{V}, 1\right) = m_0\theta^{-\alpha}$$

- Market tightness sufficient statistics for the job filling and job finding probabilities in the model

- $\Pi(s, z, n_{-1})$ denotes the value of a firm entering the period with productivity (s, z) and employment n_{-1}

- Firms choose how many vacancies v to post and current stock of employees, n, by solving the following problem

$$\Pi(s, z, n_{-1}) = \max_{n,v} \quad szF(n) - w(s, z, n)n - c_v v + \frac{1}{1+r}\tilde{\Pi}(s, z, n_{-1})$$

$$\text{and} \quad \tilde{\Pi}(s, z, n) = \sum_{z'} \Pi(s, z', n)\Gamma(z'|z)$$

where

- $w(s, z, n)$ is the wage bargained by the firm with its workers
- c_v is the cost of posting a vacancy

- Firms seek a level of employment that maximizes profits subject to a dynamic constraint on the evolution of firm's employment
- Firms face frictions that limit the rate at which vacancies may be filled.
- Since vacancy posted in a given period will be filled with probability $\phi_f < 1$ prior to production, then:

$$\underbrace{n}_{\substack{\text{new stock}\\ \text{of employees}}} = \underbrace{\phi_f v}_{\text{new hires}} + \underbrace{n_{-1}}_{\substack{\text{old stock}\\ \text{of employees}}} \quad \text{if} \quad \mathbf{1}^+[n > n_{-1}] = 1$$

- Notice:
    - No downward adjustment costs (e.g. firing costs) for firm-initiated separation
    - No worker-initiated separation

- Using the law of motion for firm-level employment, the problem of the firms becomes:

$$\Pi(s,z,n_{-1}) = \max_{n} \quad szF(n) - w(s,z,n)n - \frac{c_v}{\phi_f} \times \max\{0, n - n_{-1}\} + \frac{1}{1+r}\sum_{z'} \Pi(s,z',n)\Gamma(z'|z)$$

where $\frac{c_v}{\phi_f}$ is the (endogenous) cost of scaling employment up.

- Recall firm problem in Hopenhayn and Rogerson (1993)

$$\Pi(s,z,n_{-1}) = \max_{n} \quad szF(n) - wn - \tau \times \max\{0, n_{-1} - n\} + \frac{1}{1+r}\sum_{z'} \Pi(s,z',n)\Gamma(z'|z)$$

- Necessary (not sufficient) condition for a solution to the firm problem, conditional on $\Delta n \neq 0$

$$sz\frac{\partial F(n)}{\partial n} + \frac{1}{1+r}\sum_{z'}\frac{\partial \Pi(s,z',n)}{\partial n}\Gamma(z'|z) = \frac{\partial w(s,z,n)n}{\partial n} + \frac{c_v}{\phi_f}\mathbf{1}^{+}$$

- There is a kink in the value function around $n = n_{-1}$
  - partial irreversibility of separation decisions in the model
  - while separation is costless, it is costly to reverse such a decision because of hiring (posting vacancies) costs
  - as in Bentolila and Bertola (1990) but hiring costs endogenous

- Optimal employment policy characterized by two reservation values for firm's productivity z, $z_L(s,n_{-1})$ and $z_H(s,n_{-1})$ such that $\forall z \in (z_L(s,n_{-1}), z_H(s,n_{-1})) \implies n = n_{-1}$ (employment inaction region)

- Value of worker currently employed in a firm with productivity (s, z) and n_{-1} employees

$$J^e(s,z,n_{-1}) = w(s,z,n(s,z,n_{-1})) + \beta \tilde{J}^e(s,z,n)$$

where

$$\tilde{J}^e(s,z,n) = \sum_{z'} \underbrace{p^f(s,z',n(s,z,n_{-1}))}_{\substack{\text{endogenous} \\ \text{firing probability}}} \Gamma(z'|z) J^u +$$

$$+ \sum_{z'} (1 - p^f(s,z',n(s,z,n_{-1}))) J^e(s,z',n) \Gamma(z'|z)$$

- Value of an unemployed worker

$$J^u = b + \beta \left((1-\phi_w) J^u + \phi_w \sum_{s,z'} \int_n J^e(s,z',n) d \underbrace{\psi(s,z',n)}_{\substack{\text{endogenous} \\ \text{distribution of} \\ \text{hiring firms}}} \right)$$

or equivalently

$$J^u = \frac{b}{1-\beta} + \frac{\beta}{1-\beta} \left[\phi_w \sum_{s,z'} \int_n \underbrace{\left[J^e(s,z',n) - J^u \right]}_{\text{gain from being hired}} d\psi(s,z',n) \right]$$

- Frictions in the labor market implies makes costly for firms and workers to find alternative employment relationships.
- Quasi-rents that firm and its workers can bargain over.
- Standard search model with constant marginal product (without large firms):
  - the rents of each employment relationship are independent of the rents of all other employment relationships
  - firms can bargain with each of their workers independently
- Search model with decreasing marginal product:
  - the rents of each individual employment relationship depend on the number of workers employed!
  - the rent from "the" marginal worker lower than the rent from all infra-marginal hires due to diminishing marginal product

- Intra-firm bargaining protocol a là Stole and Zwiebel (1994)
  - generalization of the Nash solution to a setting with diminishing returns
  - Nash bargaining over the marginal surplus of firm-worker relationship

- Intuitions:
  - If the firm has only one worker, the firm and worker simply strike a Nash bargain.
  - If a second worker is added, the firm and the additional worker know that, if their negotiations break down, the firm will agree to a Nash bargain with the remaining worker. In this sense, the second employee regards herself as being on the margin.
  - By induction, then, the firm approaches negotiations with the n^{th} worker as if that worker were marginal too
  - the wage that solves the bargaining problem is that which maximizes the marginal surplus.

- Wages are set after employment has been determined
  - Hiring costs are sunk and labor market is closed setting
- Firm marginal surplus $S^f(s, z, n; w)$

$$
\begin{aligned}
S^f(s,z,n;w) &= \frac{\partial \Pi(s,z,n_{-1})}{\partial n} \\
&= sz\frac{\partial F(n)}{\partial n} - w(s,z,n) - \frac{\partial w(s,z,n)}{\partial n}n + \frac{1}{1+r}\frac{\partial \tilde{\Pi}(s,z,n)}{\partial n}
\end{aligned}
$$

- Worker marginal surplus $S^w(s, z, n; w)$

$$
S^w(s,z,n;w) = J^e(s,z,n) - J^u
$$

- Bargaining problem

$$w(s,z,n) = \arg\max_{w} \quad S^f(s,z,n;w)^{\gamma} S^w(s,z,n;w)^{1-\gamma}$$

where $\gamma \in (0,1)$ is the worker's bargaining power

- Nash splitting rule

$$S^w(s,z,n;w) = \gamma[S^f(s,z,n;w) + S^w(s,z,n;w)]$$
$$S^f(s,z,n;w) = (1-\gamma)[S^f(s,z,n;w) + S^w(s,z,n;w)]$$

- Wage solves the following ODE

$$w(s,z,n) = (1-\gamma)b + \gamma\left[sz\frac{\partial F(n)}{\partial n} - \frac{\partial w(s,z,n)}{\partial n}n + \frac{1}{1+r}c_v\phi_w\right]$$

- Wages are:
  - increasing in the worker's bargaining power
  - increasing in the marginal product of labor
  - increasing in job finding probability and marginal costs of hiring
  - increasing in home production
- Extra term: $\frac{\partial w(s,z,n)}{\partial n} n$
  - If negotiation breaks, the firm will have to pay its remaining workers a higher wage (higher marginal product)
  - Inefficient incentive to over-employ workers
- Specific solution (with $F(n) = n^\alpha$)

$$w(s,z,n) = (1-\gamma)b + \gamma \left[\frac{szn^{\alpha-1}}{1-\gamma(1-\alpha)} + \frac{1}{1+r} c_v \phi_w \right]$$

- Binmore et al. (1986) bargaining solution
  - alternating offers generalized to a setting when marginal returns are diminishing
- The threats are to extend bargaining rather than to terminate it in case of disagreement
  - disagreement payoffs determines the bargaining outcomes, not the outside option payoff
- Breakdown of negotiations generates a surplus to split between parties, which is equal to the marginal flow surplus

- Firm marginal flow surplus $S^f(s, z, n; w)$

$$S^f(s,z,n;w) = sz\frac{\partial F(n)}{\partial n} - w(s,z,n) - \frac{\partial w(s,z,n)}{\partial n}n$$

- Worker marginal flow surplus $S^w(s, z, n; w)$

$$S^w(s,z,n;w) = w(s,z,n) - b$$

- Wage solves the following ODE

$$w(s,z,n) = (1-\gamma)b + \gamma\left[sz\frac{\partial F(n)}{\partial n} - \frac{\partial w(s,z,n)}{\partial n}n\right]$$

- No influence of labor market tightness on wages (Hall and Milgrom 2008)

- Let $\mu_t(s, z, n; \theta_t)$ be the measure of firms over individual state (s, z, n) when the market tightness is θ_t at time t
- Evolution of distribution over time:

$$\mu_{t+1}(s, z, n'; \theta_{t+1}) = T_t(\mu_t(s, z, n; \theta_t), \theta_t)$$

where

$$T_t(\mu_t(s, z, n; \theta_t);\theta_t) = \sum_{s,z} \int_n \psi_t(s, z', n'|z, n; \theta_t) d\mu_t(s, z, n; \theta_t)$$

and

$$\psi_t(s, z', n'|z, n; \theta_t) = \mathbf{1}[n(s, z', n; \theta_t) = n']\Gamma(z'|z)$$

- Aggregate employment:

$$N = \sum_{s,z,n_{-1}} n(s,z,n_{-1}) d\mu(s,z,n_{-1})$$

- Total separation:

$$S = \sum_{s,z,n_{-1}} \max\{0, n_{-1} - n(s,z,n_{-1})\} d\mu(s,z,n_{-1})$$

- Total hires: $H = \phi_w U$
- Labor market dynamics: $N' = H - S + N$
- Labor resource constrain: $N + U = L$

A steady-state competitive equilibrium is a market tightness θ, a wage schedule $w(s, z, n)$, an optimal policy function for employment, $n(s, z, n_{-1})$, and a distribution $\mu(s, z, n)$, such that:

- **Firms optimality**: $n(s, z, n_{-1})$ solves the problem of the firm
- **Bargaining**: $w(s, z, n)$ are the solution to the intra-firm Nash bargaining problem
- **Stationarity**:
  - the distribution $\mu(s, z, n)$ replicates itself through productivity shocks and hiring/firing decisions
  - workers hires and separation balance each other, i.e. $\phi_w U = S$

- **Step 1**: guess market tightness θ^0
- **Step 2**: compute job finding and filling probabilities, ϕ_w^0, ϕ_f^0
- **Step 3**: compute wages, $w(s, z, n; \theta^0)$
- **Step 4**: solve the problem of the firm and obtain policy functions for employment $n(s, z, n_{-1})$
- **Step 5**: simulate the economy for a large number of firms and compute the stationary distribution, $\mu(s, z, n)$
- **Step 6**: use $\mu(s, z, n)$ to compute aggregate employment and total separation
- **Step 7**: obtain new value for market tightness, θ^1, using the stationarity condition and labor resource constraint
- **Step 8:** check convergence
    - if not achieved, use θ_1 and go back to step 2 till convergence
    - if achieved, store $\theta^* = \theta_0$

Implication 1: Inaction region

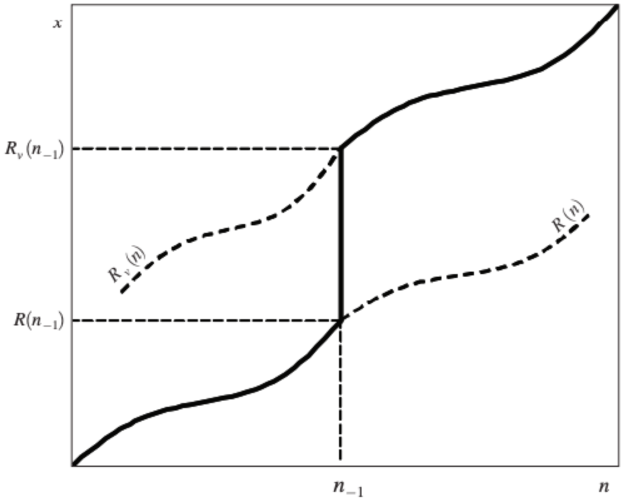


FIGURE 1. OPTIMAL EMPLOYMENT POLICY OF A FIRM

- Vacancy costs create a kink in the policy function

Implication 2: Firm size distribution

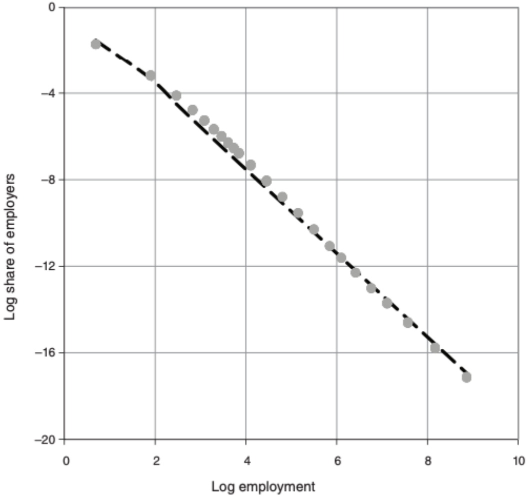


FIGURE 2. EMPLOYER SIZE DISTRIBUTION: MODEL VERSUS DATA

Notes: The dots plot data on the shares of firms in successive employment categories for the years 2002 to 2006 based on data on employment by firm-size class from the Small Business Administration. The dashed line plots the steady-state distribution of employment across firms implied by the model using the parameters reported in Table 1.

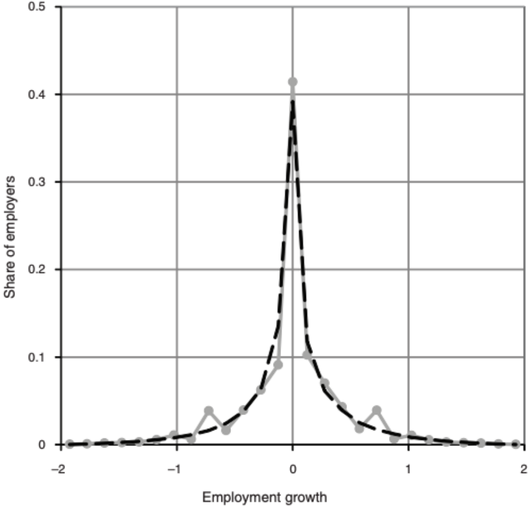


FIGURE 3. EMPLOYMENT GROWTH DISTRIBUTION: MODEL VERSUS DATA

Notes: The dotted line plots the cross-sectional distribution of employment growth based on data for continuing establishments from the Longitudinal Business Database pooled over the years 1992 to 2005. The dashed line plots the steady-state distribution of employment growth in the model using the parameters reported in Table 1.

Implication 4: Wage dispersion

- Wage dispersion between firms
  - usually accounts for 20-30% percent of overall wage dispersion
  - possible fixes:
    - workers heterogeneity
    - on-the-job search

- Counterfactual size-wage premium
  - small firms pay higher wages in the model (for some calibration)
  - two conflicting forces:
    - large firms are those more productive: wages $\uparrow$
    - large firms have lower marginal product of labor: wages $\downarrow$

Extension: Entry and Exit dynamics

- The problem of the incumbent firms now reads as follows:

$$\Pi(z, n_{-1}) = \max_{n} \quad zF(n) - w(z,n)n - c_o - \frac{c_v}{\phi_f}\max\{0, n - n_{-1}\} + \frac{1}{1+r}\max\left\{0, \sum_{z'} \Pi(z', n)\Gamma(z'|z)\right\}$$

where c_o denotes a cost of operation

- A solution to this problem is a policy function for employment $n(z, n_{-1})$ together with policy function for exit $\mathbf{1}^x(z, n_{-1})$
- There exists a thresholds $z^*(n)$ such that $\Pi(z^*(n), n) = 0$ and $\forall z < z^*(n) \implies \mathbf{1}^x(z, n_{-1}) = 1$.

- Potential entrants are ex-ante identical
- New entrants $M \geq 0$ pay c_e and enter the industry with no employees
- Draw productivity level z from $\Gamma^e(z)$ (ergodic distribution obtained from $\Gamma(z|z_{-1})$)
- Free entry condition:

$$\Pi^e = \frac{1}{1+r} \sum_z \Pi(z, 0) \Gamma^e(z) \leq c_e$$

with equality if $M > 0$.

- **Step 1**: guess market tightness θ^0
- **Step 2**: compute job finding and filling probabilities, ϕ_w^0, ϕ_f^0
- **Step 3**: compute wages, $w(s, z, n; \theta^0)$
- **Step 4**: solve the problem of the incumbent firms
- **Step 5**: compute the value of entry, Π^e and check if free-entry condition is satisfied:
  - if no, make a new guess θ^1 and go back to step 2 till convergence
  - if yes, store $\theta^* = \theta^0$
- **Step 6**: simulate the economy for a large number of firms and compute average employment and job destruction, $\bar{\ell}$ and $\bar{s}$
- **Step 7**: recover unemployment and number of firms M using stationarity condition $M\bar{s} = \phi_w U$ and labor resource constraints, $M\bar{\ell} + U = L$

Extension: Firing costs

- The problem of the firms now reads as follows:

$$\Pi(z, n_{-1}) = \max_{n} \quad zF(n) - w(z,n)n$$
$$- \frac{c_v}{\phi_f} \max\{0, n - n_{-1}\} - c_f \max\{0, n_{-1} - n\}$$
$$+ \frac{1}{1+r} \sum_{z'} \max\{0, \Pi(z', n)\} \Gamma(z'|z)$$

where c_f denotes an (exogenous) cost of scaling employment down

- Firing costs have similar implications as fixed costs. Why?

- Conditional on $\Delta n > 0$ and not-exiting

$$z\frac{\partial F(n)}{\partial n} - \frac{\partial w(z,n)n}{\partial n} + \frac{1}{1+r}\sum_{z'}\frac{\partial \Pi(z',n)}{\partial n}\Gamma(z'|z) = \frac{c_v}{\phi_f}$$

- Conditional on $\Delta n < 0$ and not-exiting

$$z\frac{\partial F(n)}{\partial n} - \frac{\partial w(z,n)n}{\partial n} + \frac{1}{1+r}\sum_{z'}\frac{\partial \Pi(z',n)}{\partial n}\Gamma(z'|z) = -c_f$$

- Optimal employment choice characterized by two reservation values for productivity z, $z_L(n_{-1})$ and $z_H(n_{-1})$ such that $\forall z \in (z_L(n_{-1}), z_H(n_{-1})) \implies n = n_{-1}$
- Notice
  - $z_L(n_{-1})$ decreases with firing costs!
  - marginal firm surplus is negative when $c_f > 0$

- Bargaining with a hiring firms

$$w^h(z,n) = \arg\max S^f(z,n;w)^\gamma S^w(z,n;w)^{1-\gamma}$$

- Bargaining with a firing firms:
  - firing firms optimally choose n s.t. $S^f(z,n;w) = -c_f < 0$
  - Nash splitting rule implies negative workers' surplus

$$S^w(z,n;w) = \frac{\gamma}{(1-\gamma)} S^f(z,n;w) < 0$$

    hence workers have incentive to quit!
  - Ensure workers' participation constraint, i.e.

$$S^w(z,n;w) \geq 0$$

  - Wage solution such that $S^w(z,n;w) = 0$, i.e.

$$w^f(z,n) = J^u - \tilde{J}^e(z,n)$$

Output loop

- **Step 1**: guess a wage schedule $w^0(z, n)$
- **Step 2**: solve the **Inner loop**
- **Step 3**: use $n(s, z, n_{-1})$ together with solution to bargaining problem to obtain a new schedule for wages $w^1(z, n)$
- **Step 4**: check convergence
  - if no, use $w^1(z, n)$ as new guess and go back to step 2 till convergence
  - if yes, store $w^*(z, n) = w^0(z, n)$
- **Step 5**: simulate the economy for a large number of firms and compute average employment and job destruction, $\bar{\ell}$ and $\bar{s}$
- **Step 6**: recover unemployment and number of firms M using stationarity condition $M\bar{s} = \phi_w U$ and labor resource constraints, $M\bar{\ell} + U = L$

Inner loop

- **Step 1**: guess market tightness θ^0
- **Step 2**: compute job finding and filling probabilities, ϕ_w^0, ϕ_f^0
- **Step 3**: solve the problem of the firm and obtain policy functions for employment $n(s, z, n_{-1})$
- **Step 4**: compute the value of entry, Π^e and check if free-entry condition is satisfied:
  - if no, make a new guess θ^1 and go back to step 2 till convergence
  - if yes, store $\theta^* = \theta^0$

Extension: Heterogeneous Agents

- Concave firm-level production function with J types of workers

$$y = z \underbrace{\left(\sum_{j=1}^{J} (s_j n_j)^{\rho} \right)^{\frac{\alpha}{\rho}}}_{F(\boldsymbol{n})}$$

  where
  - n_j denotes the number of group-j employees
  - $\boldsymbol{n} = (n_1, n_2, ..., n_J)$ is a vector of n_j
  - s_j denotes labor efficiency of group-j employees
  - ρ governs the elasticity of substitutions across workers groups
  - $\alpha \in (0, 1)$ governs the return to scale of the aggregate labor input in production

- Innate productivity differences across firms, $z \sim \Gamma(z)$

- Labor market segmented by skill groups j
- To hire workers, firms need to post separate vacancies for each type-j worker
  - Group specific hiring cost, c_v^j, per vacancy posted
- Unemployed group$-j$ workers can search for job in their market only
- Vacancies are matched to the pool of unemployed workers according to a matching technology $m_j(U_j, V_j)$
  - matching function allowed to differ across groups (e.g. different matching efficiencies, etc...)
- Exogenous group-specific job destruction rate δ_j

- The problem of an incumbent firms read as follows:

$$\Pi(z, \boldsymbol{n_{-1}}) = \max_{\{v_j\}_{j=1}^J} \quad zF(\boldsymbol{n}) - \sum_{j=1}^{J} w_j(z, \boldsymbol{n}) n_j - \sum_{j=1}^{J} c_v^j v_j + \frac{1}{1+r} \Pi(z, \boldsymbol{n})$$

subject to a low of motion for each group-j workers:

$$n_j = n_{j-1}(1-\delta_j) + \phi_f^j v_j$$

- Focus on steady-state: $\boldsymbol{n_{-1}} = \boldsymbol{n}$

- Steady state firm-level employment satisfies:

$$-c_v^j + \frac{1}{1+r}\frac{\partial \Pi(z,\boldsymbol{n})}{\partial n_j}\phi_f^j = 0$$

- Firms' marginal surplus

$$\begin{aligned} S_j^f(z,\boldsymbol{n}) &= \frac{\partial \Pi(z,\boldsymbol{n})}{\partial n_j} \\ &= \frac{1+r}{\delta_j + r}\left[z\frac{\partial F(\boldsymbol{n})}{\partial n_j} - w_j(z,\boldsymbol{n}) - \sum_{i=1}^{J}\frac{\partial w_i(z,\boldsymbol{n})}{\partial n_j}n_i\right] \end{aligned}$$

- Firms' marginal product at the optimal level of employment is equal to the expected recruitment

- A solution to the problem of the incumbent firms is a policy function for employment of each group-j workers such that:

$$\underbrace{z\frac{\partial F(\boldsymbol{n})}{\partial n_j}}_{\text{marginal product}} = \underbrace{w_j(z,\boldsymbol{n})}_{\text{wage}} + \underbrace{\sum_{i=1}^{J}\frac{\partial w_i(z,\boldsymbol{n})}{\partial n_j}n_i}_{\text{employment effect on wages}} + \underbrace{(\delta_j + r)\frac{c_v^j}{\phi_f^j}}_{\text{turnover cost}}$$

- Value of being employed

$$V_j^e(z,\boldsymbol{n}) = w_j(z,\boldsymbol{n}) + \frac{1}{1+r}\left[\delta_j V_j^u + (1-\delta_j)V_j^e(z,\boldsymbol{n})\right]$$

- Value of being unemployed

$$V_j^u = b + \frac{1}{1+r}\left[\phi_w^j \sum_z \int_{\boldsymbol{n}} V_j^e(z,\boldsymbol{n})\psi(z,\boldsymbol{n})d\boldsymbol{n} + (1-\phi_w^j)V_j^u\right]$$

- Workers' marginal surplus

$$\begin{aligned} S_j^w(z,\boldsymbol{n}) &= V_j^e(z,\boldsymbol{n}) - V_j^u = \\ &= \frac{1+r}{\delta_j + r}\left[w_j(z,\boldsymbol{n}) - \frac{r}{1+r}V_j^u\right] \end{aligned}$$

- Intra-firm bargaining problem (Stole and Zwiebel 1994) extended to heterogeneous workers
- Workers bargaining power γ_j allowed to vary across groups
- Nash sharing rule

$$\gamma_j S_j^f(z, \boldsymbol{n}) = (1-\gamma_j) S^w(z, \boldsymbol{n}) \quad \forall j = 1, 2, ...J$$

- Wage solves the following a non-linear system of ODE

$$w_j(z, \boldsymbol{n}) = \frac{r}{1+r}(1-\gamma_j)V_j^u + \gamma_j \left[z \frac{\partial F(\boldsymbol{n})}{\partial n_j} - \sum_{i=1}^{J} \frac{\partial w_i(z, \boldsymbol{n})}{\partial n_j} n_i \right] \quad \forall j$$

- Wage solution

$$w_j(z,\boldsymbol{n}) = \underbrace{\frac{r}{1+r}(1-\gamma_j)V_j^u}_{(1-\gamma_j)b+\gamma_j\theta\frac{cv}{1+r}} + \int_0^1 x^{\frac{1-\gamma_j}{\gamma_j}} z \frac{\partial F(\boldsymbol{nA_j})}{\partial n} dx$$

where the vector $\boldsymbol{nA_j}$ is equal to

$$\boldsymbol{nA_j} = \left(n_1 z^{\frac{\beta_1}{1-\beta_1}\frac{1-\beta_j}{\beta_j}}, n_2 z^{\frac{\beta_2}{1-\beta_2}\frac{1-\beta_j}{\beta_j}}, ..., n_J z^{\frac{\beta_J}{1-\beta_J}\frac{1-\beta_j}{\beta_j}} \right)$$

Extension: Aggregate shocks

- Baseline model with aggregate productivity shocks:

$$p' = \begin{cases} p + \sigma_p & \text{with probability} \quad 1/2 \\ p - \sigma_p & \text{with probability} \quad 1/2 \end{cases}$$

- Firms need to forecast future wages, hence labor market tightness, hence entire distribution of employment across firms
- Approximate evolution of aggregate employment and market tightness around their steady-state values (for $\sigma_p \approx 0$)

$$N' \approx N^* + \nu_N(N - N^*) + \nu_p(p' - p)$$
$$\theta' \approx \theta^* + \mu_N(N - N^*) + \mu_p(p' - p)$$

- Krusell-Smith algorithm:
  - iterate numerically over the parameters $(\nu_N, \nu_p, \mu_N, \mu_p)$

- **Step 1**: guess a set of coefficients $\nu_N^0, \nu_p^0, \mu_N^0, \mu_p^0$
- **Step 2**: forecasting rules gives value for (N', θ, θ') given (p, N)
- **Step 3**: solve the problem of the firm
  - state-space expanded to (p, N)
  - discreteness of p effectively implies only one more state variable

  and store policy functions for employment $n(s, z, n_{-1}, p, N)$
- **Step 4**: Simulate the economy for a larger number of firms and compute aggregate number of hires and separation in each period, $H(p, N)$ and $S(p, N)$
  - this requires to compute the distribution of employment across incumbent firms

- **Step 5**: Compute employment using the labor market law of motion:

$$N = N_{-1} + H(p, N) - S(p, N)$$

- **Step 6**: Compute equilibrium market tightness from the identity condition for hires:

$$H(p, N) = \theta(L - N)$$

- **Step 7**: Use time series to estimate the forecasting coefficient via OLS, $\nu_N^1, \nu_p^1, \mu_N^1, \mu_p^1$
- **Step 8**: Check if convergence is satisfied
    - if no, use $\nu_N^1, \nu_p^1, \mu_N^1, \mu_p^1$ as new guess and go to Step 1 until convergence
    - if yes, store $\nu_N^* = \nu_N^1, \nu_p^* = \nu_p^1, \mu_N^* = \mu_N^1, \mu_p^* = \mu_p^1$

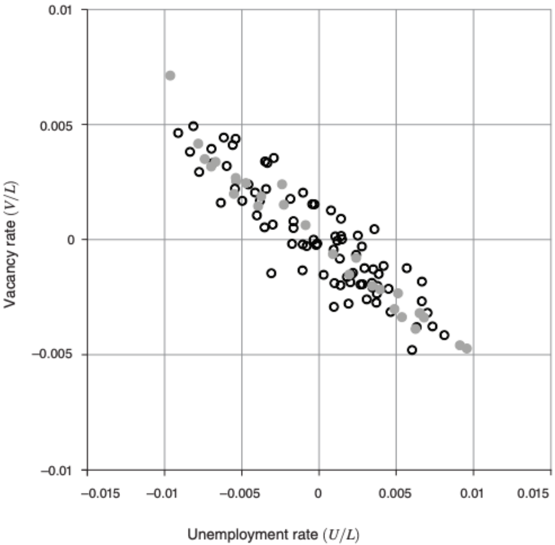


FIGURE 5. BEVERIDGE CURVE: MODEL VERSUS DATA

- Standard search model unable to generate enough cyclicality in job creation jointly with workers flow into unemployment

$$\frac{\partial \theta}{\partial p} \approx \frac{(1-\gamma)p}{m_0[(1-\gamma)(p-b) - \frac{\gamma}{1+r}c_v\theta] + \frac{\gamma}{1+r}c_v\theta}$$

- Large firms induce an amplification mechanism

$$\frac{\partial \theta}{\partial p} \approx \frac{(1-\gamma)p}{\omega m_0[(1-\gamma)(p-b) - \frac{\gamma}{1+r}c_v\theta] + \frac{\gamma}{1+r}c_v\theta}$$

where

- $\omega < 1$ is the steady-state employment share of hiring firms (always one in standard model)
- p is a a weighted average of the average and marginal flow surpluses (only average in standard model)

Implication 2: Amplification

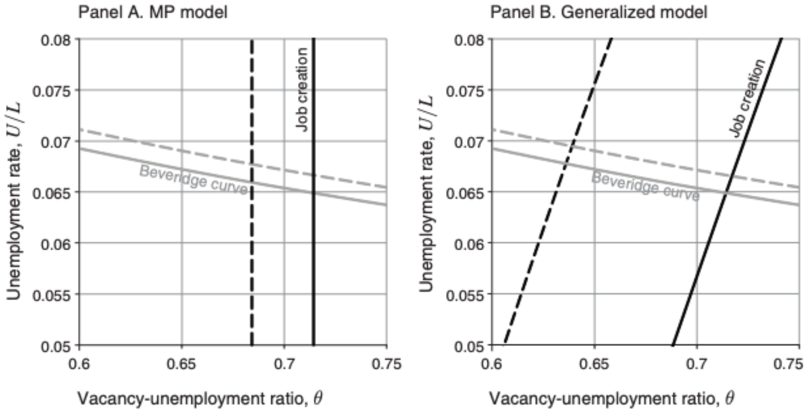


FIGURE 4. JOB CREATION AND BEVERIDGE CURVE CONDITIONS: MP MODEL VERSUS GENERALIZED MODEL

Notes: The figure illustrates the steady-state shifts in the job creation and Beveridge curve conditions in the MP model (panel A) and the generalized model (panel B) detailed in Sections I and II. The calibrations, respectively, are taken from panels B and A of Table 2. As a result, the initial steady states, and the shifts in the Beveridge curve are identical across models.

Implication 3: Propagation

- Standard search model unable to generate sluggish response of equilibrium labor market tightness to aggregate shocks to labor productivity
  - vacancy-unemployment ratio is a jump variable
- Large firms induce a propagation mechanism
  - vacancy-unemployment ratio depends on the distribution of employment across establishments
  - distribution of establishment is not a jump variable
  - sluggish behavior generated by vacancy costs and idiosyncratic shocks

Implication 3: Propagation

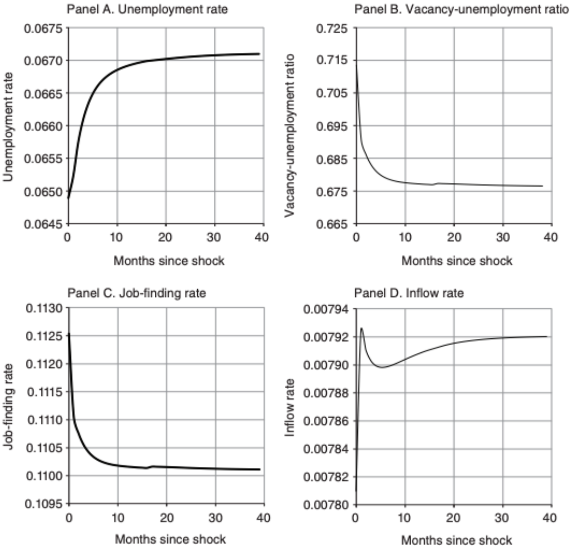


Figure 6. Model Impulse Responses to a Permanent 1 Percent Decline in Aggregate Labor Productivity, p

- Acemoglu D. and Hawkins W.B. 2014. “Search with multi-worker firms.” Theoretical Economics, Vol. 9 , No. 3, pp. 583-628
- Kaas L. and Kircher P. 2015. “Efficient firm dynamics in a frictional labor market.” American Economic Review, Vol. 105, No. 10, pp. 3030-60.

1. Lecture 1: Firm dynamics with frictional labor market
2. Lecture 2: Firm growth with on-the-job search
3. Lecture 3: Structural estimation and identification

- Fajgelbaum P. 2020. "Labour Market Frictions, Firm Growth, and International Trade." Review of Economic Studies, Vol. 87, No. 3, pp. 1213–1260
  - Multi-worker heterogeneous firms
  - Search frictions with on-the-job search
  - Wage settings
    - Nash bargaining between firm and unemployed worker
    - Bertrand competition between firms to poach employer worker
  - Life-cycle wage growth
  - Endogenous *misallocation* of labor across firms
- Extensions:
  - Ex-ante heterogeneous workers
  - Dynamic investment decisions

- Labor market frictions prevent labor reallocation across firms
  - lower firm growth
  - higher resource misallocation
- Firms grow by poaching workers
  - Employees move directly to new employers
- Goal: Combine firm dynamics with on-the-job search
- How would changes in search frictions affect firm growth?

- Time is continuous
- Exogenous measure of workers
  - stochastic life-cycle
    - δ_w: workers retirement rate
  - ex-ante homogeneous
  - employed/unemployed
  - off- and on-the-job search
- Endogenous measure of firms
  - stochastic life-cycle
    - δ_f: exogenous firms' exit rate
  - heterogeneity in productivity
  - entry/exit dynamics
  - linear production function
  - firm size bounded by convex recruiting costs

- Firm-level production technology

$$y = \int_0^{\ell} g(i, z)\psi^e(i|z)di$$

where $\psi^e(i|z)$ denotes the share of worker i in a firm z with total workforce ℓ

- Firm-worker match production:

$$g(i, z) = Az = g(z) \quad \forall i$$

where A is an aggregate shifter

- Given workers homogeneity, technology can be expressed as a linear function of total workforce

$$y = Az\ell$$

- Search frictions with on-the-job search
  - workers search for jobs at different (exogenous) intensity
  - firms search for workers at different (endogenous) intensity

- Total pool of searching workers :

$$\tilde{\lambda}_u u + \tilde{\lambda}_e (1-u)$$

  where $\tilde{\lambda}_i$ is the search intensity (visibility) of group $i = u, e$

- Total pool of searching firms:

$$M\bar{s}$$

  where M is the measure of active firms while $\bar{s}$ is the average search effort

- CRS matching function: $m(M\bar{s}, \tilde{\lambda}_u u + \tilde{\lambda}_e (1-u))$

- Contact rate for workers in group $i = u, e$

$$\lambda_i = \tilde{\lambda}_i \frac{m(M\bar{s}, \tilde{\lambda}_u u + \tilde{\lambda}_e(1-u))}{\tilde{\lambda}_u u + \tilde{\lambda}_e(1-u)}$$

- By homogeneity of degree 1:

$$\lambda_i = \tilde{\lambda}_i \chi \left(\frac{M\bar{s}}{\tilde{\lambda}_u u + \tilde{\lambda}_e(1-u))} \right)$$

where $\chi(x) = m(1, x)$

- Labor market tightness

$$\theta = \frac{M\bar{s}}{\tilde{\lambda}_u u + \tilde{\lambda}_e(1-u))}$$

- Total number of matches

$$m = \lambda_u u + \lambda_e (1-u)$$

- Total number of matches of a firm exerting $s(z)$ effort

$$m(z) = [\lambda_u u + \lambda_e (1-u)] \frac{s(z)}{M\bar{s}}$$

- Rate at which firms contact workers from unemployment

$$q_u = \frac{m(z)}{s(z)} \frac{\lambda_u u}{\lambda_u u + \lambda_e (1-u)} = \frac{\lambda_u u}{M\bar{s}}$$

- Rate at which firms contact workers from employment

$$q_e = \frac{m(z)}{s(z)} \frac{\lambda_e (1-u)}{\lambda_u u + \lambda_e (1-u)} = \frac{\lambda_e (1-u)}{M\bar{s}}$$

- Postel-Vinay and Robin (2002): firms observe the current status of the worker, tender take-it-or-leave-it wage offers, and commit to the value promised
  - the outcome similar to Bertrand competition
  - the firm offering the job of greater total value obtains the worker, offering in exchange a value equal to what the worker could obtain in the alternative employment
- Cahuc et al. (2006): the worker additionally splits the surplus with the higher-value firm according to the conventional Nash solution rule
  - total value in the lower-value job used as outside option
- In both settings a worker is hired under a flat wage profile until leaving or triggering a renegotiation.

- Flinn and Mullins (2016), Flinn et al. (2017): employers are not able to commit to wage offers, or is not able to verify claims that the individual has received an competing offers.
  - the outside option in the wage determination problem remains the value of unemployed search
  - moving to unemployment is the only action available to the employee at any moment in time
  - no wage renegotiation triggered within job
  - wage gains only through job-to-job mobility
- In this last setting a worker is hired under a flat wage profile until leaving

- Bertrand competition + wage bargaining (Cahuc et al., 2006)
  - collapses to Mortensen-Pissarides' bargaining when $\lambda_e = 0$

- $\beta \in (0,1)$: workers bargaining power

- Negotiations with the unemployed
  - Split the surplus according to Nash bargaining
  - Negotiate wage w_u such that:

$$\underbrace{W(w_u, z)}_{\substack{\text{value accruing}\\\text{to workers}}} = \beta \underbrace{V(z)}_{\substack{\text{value of}\\\text{the match}}}$$

- Notice: value of the match $V(z)$ independent of wages w!

- Renegotiation with the employed
  - happens only when either side has an interest to separate if they do not obtain an improved offer
  - on-the-job search generates alternative opportunities for workers triggering either job mobility or responses to the outside offers
- Match values alternative employers: $V(z')$
- Three scenarios:
  - $V(z') > V(z)$: the worker moves to the alternative job
  - $W(w,z) < V(z') < V(z)$: worker uses the outside offer to negotiate up her wage
  - $V(z') < W(w,z)$: the worker has nothing to gain from the competition between z and z'

- Scenario 1: the worker moves to the alternative jobs and uses the previous match value as the outside option when bargaining
  - Let $\mathcal{A}(z) := \{z' \in \mathcal{Z} \quad : V(z') > V(z)\}$
  - Negotiate w_e such that:

$$W(w_e(z, z'), z') = \beta V(z') + (1 - \beta)V(z)$$

- Scenario 2: the worker doesn't move and uses the outside offer as the option value when bargaining
  - Let $\mathcal{B}(z) := \{z' \in \mathcal{Z} : V(z') < V(z) < W(w, z)\}$
  - Negotiate w_e such that:

$$W(w_e(z, z'), z) = \beta V(z) + (1 - \beta)V(z')$$

- Scenario 3: nothing happens

- Value of an unemployed worker:

$$(r+\delta_w)U = b + \lambda_u \int_z \max\{0, \underbrace{W(w_u(z),z) - U}_{\text{gains from UtE movements}}\} dP(z)$$

- Value of a worker employed in a firm z at a wage z

$$\begin{aligned}(r+\delta_w)W(w,z) = w + \delta_f & \underbrace{(U - W(w,z))}_{\text{losses from separation}} \\ & + \lambda_e \int_{z' \in \mathcal{A}(z)} \underbrace{(W(w_e(z,z'),z) - W(w,z))dP(z')}_{\text{gains from re-negotiation}} \\ & + \lambda_e \int_{z' \in \mathcal{B}(z)} \underbrace{(W(w_e(z,z'),z') - W(w,z))}_{\text{gains from EtE movements}} dP(z')\end{aligned}$$

- Value for firm with productivity z with worker employed at wage w

$$(r+\delta_w)J(w,z) = g(z) - w + \delta_f \underbrace{(0 - J(w,z))}_{\text{losses from separation}}$$
$$+ \lambda_e \int_{z' \in \mathcal{A}(z)} \underbrace{(J(w_e(z,z'),z) - J(w,z))}_{\text{losses from re-negotiation}} dP(z')$$
$$+ \lambda_e \int_{z' \in \mathcal{B}(z)} \underbrace{(0 - J(w,z))}_{\text{losses from JtJ movements}} dP(z')$$

Using the implicit solutions for wages:

$$(r+\delta_w)U = b + \beta\lambda_u \int_z \max\{0, V(z)-U\} dP(z)$$

$$\begin{aligned}(r+\delta_w+\delta_f)W(w,z) &= w + \delta_f U \\ &+ \lambda_e \int_{z'\in\mathcal{A}(z)} ((1-\beta)V(z') - \beta V(z) - W(w,z)) dP(z') \\ &+ \lambda_e \int_{z'\in\mathcal{B}(z)} ((1-\beta)V(z) - \beta V(z') - W(w,z)) dP(z')\end{aligned}$$

$$\begin{aligned}(r+\delta_w+\delta_f)J(w,z) &= g(z) - w \\ &+ \lambda_e \int_{z'\in\mathcal{A}(z)} ((1-\beta)(V(z) - V(z')) - J(w,z)) dP(z') \\ &+ \lambda_e \int_{z'\in\mathcal{B}(z)} (-J(w,z)) dP(z')\end{aligned}$$

- Value of match $V(z) = W(w,z) + J(w,z)$ between firm with productivity z and worker employed at wage w:

$$(r + \delta_w)V(z) = g(z) + \delta_f(U - V(z)) + \lambda_e \beta \int_{z' \in \mathcal{B}(z)} (V(z') - V(z)) dP(z')$$

- Value of a match is independent of wages!
  - Re-negotiation within the firm doesn't change the total value of the match, only triggers its redistribution
- When $\lambda_e \beta = 0$, the equilibrium distribution of jobs from which workers sample, $P(z)$, does not impact the value of a match

- Value of a new worker hired from unemployment

$$S_u^f(z) = (1-\beta)(V(z) - U)$$

- Value of a new worker poached from another employer

$$S_e^f(z, z') = (1-\beta)(V(z') - V(z))$$

- Expected value of a new worker

$$S^f(z) = q_u S_u^f(z) + q_e \int_{z' \in \mathcal{C}(z)} S_e^f(z, z') dG(z')$$

  where $G(z')$ is the distribution of employment across firms

- On-the-job search expands the rate for firms by a factor of

$$q_e/q_u = \frac{\lambda_e(1-u)}{\lambda_u u}$$

- Present discounted value of profits generated by all workers who are hired by a firm with productivity z

$$\pi(z) = \max_{s} \quad S^f(z)s - c(s)$$

where
  - s denotes search/recruiting effort exerted by the firm
  - $c(s)$ is cost of search, increasing and convex in s

- Optimal search effort: $s(z) = (c')^{-1}(S^f(z))$
- Number of workers arriving at a firm z (new hires)

$$h(z) = (q_u + q_e)s(z)$$

- Firms may speed up growth by increasing recruitment effort

- Discounted sum of per-period aggregate profits

$$\Pi(z) = \int_0^\infty \pi(z) e^{-(r+\delta_f)t} dt = \frac{\pi(z)}{r + \delta_f}$$

- Number of entrants $M^e \geq 0$
- Free entry condition

$$\Pi^e = \int_{\underline{z}}^{\bar{z}} \max\{0, \Pi(z)\} d\Gamma(z) \leq c_e$$

where $\Gamma(z)$ is a CDF for firm-level productivity $z \in [\underline{z}, \bar{z}]$

- Entry decision: $\mathbf{1}^e(z) = \begin{cases} 1 & \text{if} \quad \Pi(z) > 0 \\ 0 & \text{otherwise} \end{cases}$

- Evolution of firm size (conditional on not exiting)

$$N'(z) = \underbrace{h(z)}_{\text{new hired workers}} - \underbrace{N(z)[\delta_w + \lambda_e(1 - P(z))]}_{\text{separated workers}}$$

- Dynamics of unemployment

$$du = \delta_f(1 - u) + \delta_w - (\lambda_u + \delta_w)u$$

where in steady-state: $du = 0$

- Cumulative distribution of vacant jobs across employers

$$P(z^*) = \int_{\underline{z}}^{z^*} \frac{s(z)}{\bar{s}} d\Gamma(z)$$

where $\bar{s} = \int_{\underline{z}}^{\bar{z}} s(z) d\Gamma(z)$ and $P(\bar{z}) = 1$

- Cumulative distribution employment across employers

$$\begin{aligned}(1-u)dG(z) &= u\lambda_u P(z) + (1-u)\lambda_e P(z)G(z) \\ &\quad - (1-u)(\delta_w + \delta_f + \lambda_e(1-P(z))G(z)\end{aligned}$$

where in steady-state: $dG(z) = 0$

A steady-state competitive equilibrium consists of a value function $V(z)$, contact rates λ_u, λ_e, unemployment rate u, measure of firms M, employment and sampling distributions, $G(z)$ and $P(z)$, s.t.:

- **Optimality**: the value of a match $V(z)$ attains its maximum
- **Free-entry**: $\Pi^e = c_e$
- **Stationarity**: employment and sampling distributions, $G(z)$ and $P(z)$, replicate themselves over time through firms' hiring decisions and workers' mobility decisions, and unemployment is equal to

$$u = \frac{\delta_f + \delta_w}{(\lambda_u + \delta_f + \delta_w)}$$

- **Labor market clearing**: λ_e and λ_u are consistent with workers flows in- and out-of-employment, job-to-job movements and firms' hiring decisions

Output loop

- **Step 1**: Make a guess for workers contact rates, λ_u^0 and λ_e^0, and the unemployment rate, u^0
- **Step 2**: Solve the **Inner loop** to obtain firm contact rate q_u^* and optimal search effort $s^*(z)$
- **Step 3**: Construct the average search effort $\bar{s}^*$ and use q_u^* to back out the equilibrium number of firms M^*, i.e.

$$M^* = \frac{\lambda_u^0 u^0}{\bar{s}^* q_u^*}$$

- **Step 4**: Obtain new guesses for workers contact rates, λ_u^1 and λ_e^1, and unemployment rate, u^1, using $\bar{S}$, M^*, the definition of matching function and the stationarity condition in the labor market
- **Step 5**: Iterate till convergence

Inner loop

- **Step 1**: Guess firms' contact rate with unemployed, q_u^0 and distribution of search effort, $P^0(z)$. Construct distribution of employment, $G^0(z)$
- **Step 2**: Solve for the value of the match $V(z)$ and value of unemployment U
- **Step 3**: Solve the problem of the firms and compute the optimal the search effort, $s^*(z)$
- **Step 4**: Construct discounted value of profits of new hires, $\pi(z)$
- **Step 5**: Jointly update the guesses till convergence:
  - **Step 5.1**: compute the value of entry, Π^e and check if free-entry condition is satisfied:
    - if no, make a new guess q_u^1 and go back to step 2
    - if yes, store $q_u^* = q_u^0$
  - **Step 5.2**: use the optimal the search effort $s(z)$ to update $P^1(z)$ and got back to step 2. Store $P^*(z)$ once converged.

- Given λ_e, changes in the contact rate from unemployment λ_u impact the employment allocation $n(z)$ only through general-equilibrium adjustment in U

- Suppose the value U is not an active margin:
  - pinned down by an outside sector
  - no bargaining power for worker, $\beta = 0$

  then λ_u has no impact on firm growth: changes in λ_u don't affect q_u which is pinned down by the free-entry condition

- λ_u scales the size of the employment pool, but it does not alter its composition
  - easier to hire from employment (through a larger employment pool)
  - easier to hire from unemployment (through a higher job filling rate with the unemployed)

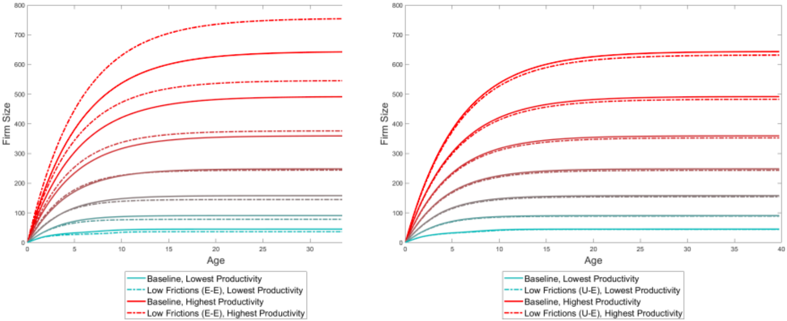
- Given λ_u, changes in the contact rate from employment λ_e impact the employment allocation $n(z)$ regardless U
- Because workers transit to high-value jobs, a higher rate of contact on the job, q_e, speeds up transitions
  - employment distribution becomes skewed towards more productive
- Firms' meeting rate with the employed not affected by free entry:

$$q_e/q_u = \frac{\lambda_e}{\lambda_u}\frac{(1-u)}{u} = \frac{\tilde{\lambda}_e}{\tilde{\lambda}_u}\frac{\frac{\lambda_u}{\lambda_u+\delta_f+\delta_w}}{\frac{\delta_f+\delta_w}{\lambda_u+\delta_f+\delta_w}} = \underbrace{\frac{\tilde{\lambda}_e}{\tilde{\lambda}_u}\lambda_u}_{\lambda_e}\frac{\frac{1}{\lambda_u+\delta_f+\delta_w}}{\frac{\delta_f+\delta_w}{\lambda_u+\delta_f+\delta_w}} = \frac{\lambda_e}{\delta_f+\delta_w}$$

Figure 2: Impact of 50% Reduction in Frictions on Firm Growth

(a) Firm Size by Age (Only E-E)

(b) Firm Size by Age (Only U-E)



Implication: Firm size distribution

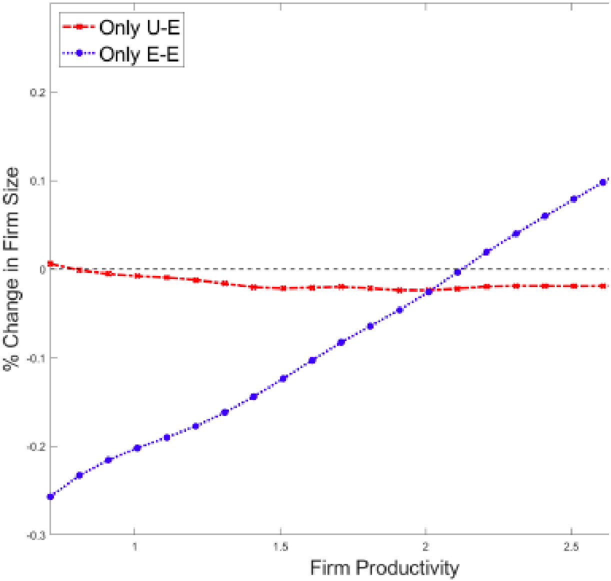
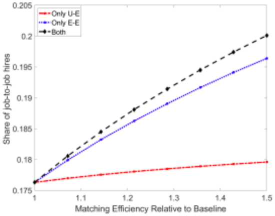


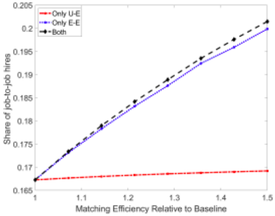
Figure 1: Impact of Lowering Frictions

Implication: Aggregates

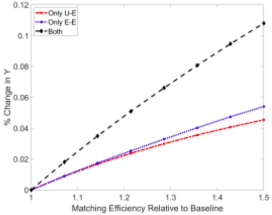
(e) Share of Job-to-job hires ($\beta = 0$)



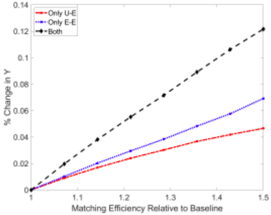
(f) Share of Job-to-job hires ($\beta = 0.44$)



(g) Income Per Capita ($\beta = 0$)



(h) Income Per Capita ($\beta = 0.44$)



Extension: Ex-ante heterogeneous workers

- Workers heterogeneous in innate skills $h(i)$
- Firm-level production technology

$$y = \int_0^\ell g(i,z)dG(i|z)$$

where $G(i|z)$ is the CDF of worker i within firm z with size ℓ

- Firm-worker match production:

$$g(i,z) = Azh(i)$$

- Technology is a linear function of total workforce and average human capital:

$$y = Az\bar{h}\ell$$

where $h(i) = \int_0^1 h(i)dG(i|z)$

- Value of unemployment for a worker i

$$(r+\delta_w)U(i) = b + \beta\lambda_u \int_z \max\{0, V(i,z) - U(i)\} dP(z)$$

- Value of match $V(i,z) = W(w,i,z) + J(w,i,z)$ between firm with productivity z and worker i employed at wage w:

$$\begin{aligned}(r+\delta_w)V(i,z) &= g(i,z) + \delta_f(U(i) - V(i,z)) \\ &\quad + \lambda_e\beta \int_{z' \in \mathcal{B}(i,z)} (V(i,z') - V(i,z)) dP(z')\end{aligned}$$

- Value of a new worker i hired from unemployment

$$S_u^f(i,z) = (1-\beta)(V(i,z) - U(i))$$

- Value of a new worker i poached from another employer z'

$$S_e^f(i,z,z') = (1-\beta)(V(i,z) - V(i,z'))$$

- Expected value of a new worker

$$S^f(z) = q_u \int_i S_u^f(i,z)dH(i) + q_e \int_{i,z' \in \mathcal{C}(z)} S_e^f(i,z,z')dG(i,z')$$

where

- $G(i,z')$ is the distribution of employed across states
- $H(i)$ is the distribution of unemployed across states

Outer loop: As before

Inner loop

- **Step 1**: Guess firms' contact rate with unemployed, q_u^0 and distribution of search effort, $P^0(z)$.
- **Step 2.1**: Solve for the value of the match $V(i,z)$ and value of unemployment $U(i)$
- **Step 2.2**: Simulate the economy for a large number of workers to construct $G(i,z)$ and $H(i)$
- **Step 3**: As before
- **Step 4**: As before
- **Step 5**: As before

Extension: Dynamic Investment Decisions

- Production linear function of total workforce

$$y = Az\ell$$

- Two technologies available: $A_1 > A_0$
    - per-period sunk cost c_x of accessing more productive technology
- Firms values changes along its life-cycle. It depends on:
    - productivity z
    - time a firm plans to wait before adopting a technology
    - whether firm has adopted a new technology or not
- Match-specific state variables:
    - productivity, z
    - time of investment, h
    - age, a

- Value of match between a worker and a firm with productivity z, age a and investing after $h(z)$ periods from entry:

$$(r+\delta_w)V(z,h,a) = A_0 z + \delta_f(U - V(z,h,a)) + \lambda_e\beta \int_{z',h',a' \in \mathcal{B}(z,h,a)} (V(z',h',a') - V(z,h,a))dP(z',h',a') + \frac{\partial V(z,h,a)}{\partial a}$$

- Value of match between a worker and a firm with productivity z, age a that already invested at $h(z)$:

$$(r+\delta_w)V(z,h) = A_1 z + \delta_f(U - V(z,h)) + \lambda_e\beta \int_{z',h',a' \in \mathcal{B}(z,h)} (V(z',h',a') - V(z,h))dP(z',h',a')$$

- Discounted sum of per-period aggregate profits

$$\Pi(z) = \max_{h \geq 0} \quad \int_0^h \pi(z,h,a) e^{-(r+\delta_f)a} da + e^{-(r+\delta_f)h} \left[\frac{\pi(z,h) - c_x}{r+\delta_f} \right]$$

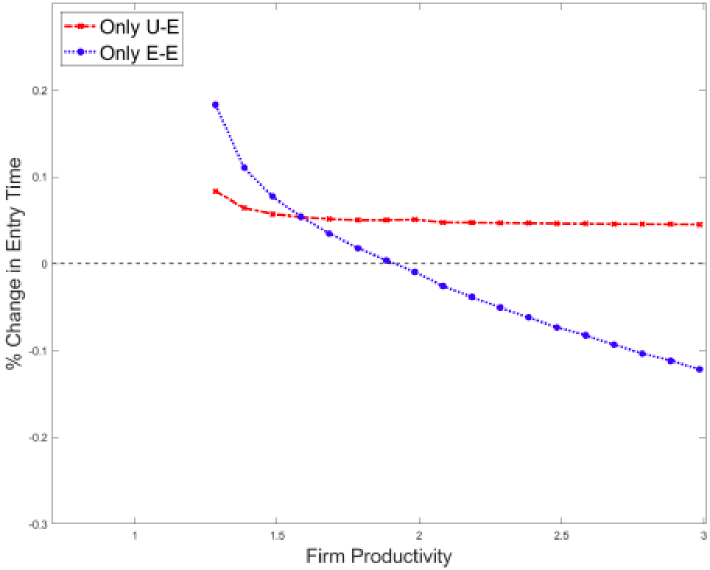
- Optimality condition:

$$\underbrace{\int_0^{h^*} e^{-(r+\delta_f)x} \frac{\partial \pi(z,h^*,x)}{\partial V(z,h^*,x)} \frac{\partial V(z,h^*,x)}{\partial x} dx}_{\text{opportunity cost of delaying investment}} \leq \underbrace{c_x}_{\substack{\text{savings from} \\ \text{delaying investment}}}$$

with equality if h^* is finite

- Notice that $h^* > 0$ since $c_x > 0$.

Implication: Time of investment



- Elsby M. and Gottfries A. 2021. "Firm dynamics, on-the-job search and labor market fluctuations." Review of Economic Studies, forthcoming.
- Bilal A.G., Engbom N., Mongey, S. and Violante, G.L. 2019. "Firm and worker dynamics in a frictional labor market." NBER working paper No. 26547.

1. Lecture 1: Firm dynamics with frictional labor market
2. Lecture 2: Firm growth with on-the-job search
3. Lecture 3: Structural estimation and identification

Lecture 3: Structural estimation

- Two estimations techniques:
  - Method of simulated moments (Gourieroux and Monfort, 1996)
    - Application: "Firms, policies, informality and the labor market" (Cisneros-Acevedo and Ruggieri, 2022)
  - Quasi-Bayesian MSM (Chernozhukov and Hong, 2003)
    - Application: "Misallocation and Inequality" (Guner and Ruggieri, 2022)

- Identification:
  - sensitivity of parameter estimates to estimation moments (Andrews, Gentzkow and Shapiro, 2017)
  - sensitivity of parameter estimates to external parameters (Jorgensen, 2020)

Structural estimation

- Let $\theta \in \Theta$ be the vector of parameters to estimate
- Let $\bar{\mathbf{m}}$ be a vector of sample statistics that our model is designed to explain
- Let $\mathbf{m}(\theta)$ be the vector of model-based counterparts to these sample statistics
- The MSM estimator is given by

$$\hat{\theta} = \arg\min \mathcal{L}(\theta)$$

where

$$\mathcal{L}(\theta) = (\mathbf{m}(\theta) - \bar{\mathbf{m}})' \hat{\mathbf{W}} (\mathbf{m}(\theta) - \bar{\mathbf{m}})$$

and $\hat{\mathbf{W}}$ is a semi positive definite matrix.

- Standard asymptotic variance (Delta method):

$$\text{var}[\hat{\theta}] \approx (\mathbf{J}'\hat{\mathbf{W}}\mathbf{J})^{-1}\mathbf{J}'\hat{\mathbf{W}}\hat{\mathbf{Q}}\hat{\mathbf{W}}\mathbf{J}(\mathbf{J}'\mathbf{W}\mathbf{J})^{-1}$$

  where
  - $\mathbf{J} = \partial(\mathbf{m}(\theta) - \bar{\mathbf{m}})/\partial\theta$
  - $\hat{\mathbf{Q}} = \hat{\text{cov}}[\mathbf{m}(\theta) - \bar{\mathbf{m}}]$

- Using $\hat{\mathbf{W}} = \hat{\text{var}}[\bar{\mathbf{m}}]^{-1}$ leads to the efficient var-cov matrix

- Hint: Setting off-diagonal terms to zero improves the stability of our estimator while maintaining consistency and keeping it independent of units of measurement (Lee and Wolpin 2006, Cosar et al 2016)

- **Step 1**: Use your data to compute $\bar{\mathbf{m}}$.
- **Step 2**: Bootstrap your data to compute $\hat{\text{var}}[\bar{\mathbf{m}}]$. Set $\hat{\mathbf{W}} = \hat{\text{var}}[\bar{\mathbf{m}}]^{-1}$
- **Step 3**: Guess a vector of parameter θ_0.
- **Step 4**: Solve and simulate your model. Construct $\mathbf{m}(\theta_0)$
- **Step 5**: Evaluate the objective function

$$\mathcal{L}(\theta_0) = (\mathbf{m}(\theta_\mathbf{0}) - \bar{\mathbf{m}})'\hat{\mathbf{W}}(\mathbf{m}(\theta_\mathbf{0}) - \bar{\mathbf{m}})$$

- **Step 6**: Search over the parametric space Θ to minimize $\mathcal{L}(\theta)$. Update θ_0 and go back to step 4.

- The function $\mathcal{L}$ can be highly non-convex, almost everywhere flat, and could have numerous discontinuity and local optima
- Alternative: Laplace type estimators (LTE) or Quasi-Bayesian estimators
  - Bayesian estimators that uses general statistical criteria in place of parametric likelihood function
  - No course of dimensionality, no need to search over parametric space
  - It allows to incorporate prior information

- Let $\theta \in \Theta$ be the vector of parameters to estimate
- Let $\pi(\theta)$ a prior density function on θ
- Quasi-posterior density function over parameters of interest:

$$p(\theta) = \frac{\exp^{\mathcal{L}(\theta)} \pi(\theta)}{\int \exp^{\mathcal{L}(\theta)} \pi(\theta) d\theta}$$

where $\mathcal{L}(\theta) = (\mathbf{m}(\theta) - \bar{\mathbf{m}})' \hat{\mathbf{W}} (\mathbf{m}(\theta) - \bar{\mathbf{m}})$

- The Quasi-Bayesian MSM estimator is defined as

$$\hat{\theta} = \int_{\theta \in \Theta} \theta p(\theta) d\theta = \int_{\theta \in \Theta} \theta \frac{\exp^{\mathcal{L}(\theta)} \pi(\theta)}{\int \exp^{\mathcal{L}(\theta)} \pi(\theta) d\theta} d\theta$$

- We use Monte-Carlo Markov Chain (MCMC) method to simulate a chain of parameters, $\{\theta_j\}_{j=1}^{N}$
- A point estimate for the parameters are obtained as the average of N_S elements of the converged MCMC chain

$$\hat{\theta} = \frac{1}{N_S} \sum_{j=1}^{N_S} \theta_j$$

- The variance for the estimates can be constructed as

$$\mathrm{var}[\hat{\theta}] = \frac{1}{N_S - 1} \sum_{j=1}^{N_S} (\theta_j - \hat{\theta})^2$$

- **Step 1**: Use your data to compute $\bar{\mathbf{m}}$.
- **Step 2**: Bootstrap your data to compute $\widehat{\text{var}}[\bar{\mathbf{m}}]$. Set $\hat{\mathbf{W}} = \widehat{\text{var}}[\bar{\mathbf{m}}]^{-1}$
- **Step 3**: Choose a starting vector of parameter θ_0.
- **Step 4**: Extract a new vector of parameters, ϕ from a proposal density function $q(\phi|\theta_j)$
- **Step 5**: Update θ^{j+1} from θ_j using the following rule

$$\theta^{j+1} = \begin{cases} \phi & \text{with prob.} \quad d(\theta^j, \phi) \\ \theta^j & \text{with prob.} \quad 1 - d(\theta^j, \phi) \end{cases}$$

where

$$d(\theta^j, \phi) = \min\left(1, \frac{\exp^{\mathcal{L}(\phi)} \pi(\phi) q(\theta^j|\phi)}{\exp^{\mathcal{L}(\theta^j)} \pi(\theta^j) q(\phi|\theta^j)}\right)$$

- **Step 6**: Repeat his procedure many times to obtain a chain of length N_S that represents the ergodic quasi-posterior distribution of θ.

- Hint: Choosing the prior $\pi(\theta)$ to be uniform and the proposal density to be a random walk, i.e. $q(\theta^j|\phi) = q(\phi|\theta^j)$, it results in the simple updating rule

$$d(\theta^j, \phi) = \min\left(1, \exp^{\mathcal{L}(\phi) - \mathcal{L}(\theta^j)}\right)$$

- Simulate 100 chains in parallel, each of length 10,000, and use the last 1,000 elements (pooled over the 100 chains) to obtain parameter estimates and the standard errors

Identification

Andrews, Gentzkow, and Shapiro (2017)

- How do targeted moments drive the identification of parameters?
- Measure of parameters θ's sensitivity to moment conditions:

$$\Lambda = -(\mathbf{J}'\hat{\mathbf{W}}\mathbf{J})^{-1}\mathbf{J}'\hat{\mathbf{W}}$$

  where
  - $\mathbf{J} = \partial(\mathbf{m}(\theta) - \bar{\mathbf{m}})'/\partial\theta$
  - $\hat{\mathbf{W}}$ is a positive semi-definite weighting matrix
- Intuitively, Λ is a local approximation to the mapping from moments to estimated parameters, i.e.

$$E[\theta] \approx \Lambda E[\mathbf{m}(\theta) - \bar{\mathbf{m}}]$$

Jorgensen (2022)

- How do estimates depend on estimation assumptions?

- Measure of parameters θ's sensitivity to external parameters, ω

$$S = \Lambda \mathbf{D} = -(\mathbf{J}'\hat{\mathbf{W}}\mathbf{J})^{-1}\mathbf{J}'\hat{\mathbf{W}}\mathbf{D}$$

  where
  - Λ is Andrews et al. (2018) sensitivity measure
  - $\mathbf{D} = \partial(\mathbf{m}(\theta) - \bar{\mathbf{m}})'/\partial\omega$

- Elasticity of the k^{th} estimated parameter to the ℓ^{th} external parameter:

$$\mathcal{E}_{k\ell} = S_{k\ell}\omega_\ell/\theta_k$$